GROWING ANNUITIES

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An article in the Journal of Financial Education by Richard Taylor [1] provided a closed-form formula for the future value of a growing annuity. This note builds on Taylor's work to provide the closed-form formula for the present value of an increasing annuity, as well as the special case formulas required when the growth rate in the annuity equals the nominal interest rate per period. In addition, the Gordon common stock valuation model is shown to be simply a special case of the present value of a growing ordinary annuity.

Annuity: A series of equal payments or receipts occurring over a specified number of periods. In an ordinary annuity, payments or receipts occur at the **end** of each period; in an annuity due, payments or receipts occur at the **beginning** of each period.

Growing Annuity: A series of payments or receipts occurring over a specified number of periods that increase each period at a constant percentage. In a growing ordinary annuity, payments or receipts occur at the **end** of each period; in a growing annuity due, payments or receipts occur at the **beginning** of each period.

The usual discussion of annuities considers level payment or receipt patterns. Formulas, as well as tables of interest factors, for dealing with such situations are well known. However, because of inflation, rising costs, and/or increasing benefits, many annuities are **not** zero growth investment or payment situations. Over time, cash flow patterns tend to grow. The following **not so well-known** formulas will quickly furnish the future value or present value of such **growing annuities.**

FUTURE VALUE OF A GROWING ORDINARY ANNUITY

The future value of a growing ordinary annuity (FVGA) answers questions like the following: "If R_1 dollars, increasing each year at an annual rate *g*, are deposited in an account at the end of each year for *n* years, and if the deposits earn interest rate *i* compounded annually, what will be the value of the account at the end of *n* years?"

$$FVGA = R_1\,(FVIFGA_{i,n,g}) \qquad (1)$$

$$FVIFGA = \frac{(1+i)^n - (1+g)^n}{(i-g)} \qquad \text{for } i \neq g \qquad (2)$$

$$= n(1+i)^{n-1} \qquad \text{for } i = g \qquad (3)$$

where

FVIFGA = future value interest factor for a growing ordinary annuity;[1]

i = the nominal interest rate per period;

n = the number of periods;

g = the periodic growth rate in the annuity;

R_1 = the receipt or payment **at the end of period 1**.

To illustrate, suppose Ms. Investor receives a 3-year ordinary annuity that begins at $1,000 but increases at a 10% annual rate. She deposits the money in a savings account at the end of each year. The account earns interest at a rate of 6% compounded annually. How much will her account be worth at the end of the 3-year period? Figure 1 provides the answer

FIGURE 1
TIME LINE FOR THE FUTURE VALUE OF A GROWING ORDINARY ANNUITY
(R_1 = $1,000; i = 6%; n = 3; and g = 10%)

End of Year

0 | 1 | 2 | 3

$1,000 (1 + .10)$1,000 $(1 + .10)^2$ $1,000

Earns no interest

$1,210.00

Compounded 1 period

1,166.00

Compounded 2 periods

1,123.60

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Future Value of a Growing Ordinary Annuity (FVGA)= $3,499.60

Or, according to Equations (1) and (2)

$$\text{FVGA} = R_1\,(\text{FVIFGA}_{6\%,3,10\%})$$

$$= (\$1{,}000)\ \frac{(1 + .06)^3 - (1 + .10)^3}{(.06 - .10)}$$

$$= (\$1{,}000)\ (3.4996) = \underline{\underline{\$3{,}499.60}}$$

PRESENT VALUE OF A GROWING ORDINARY ANNUITY

The present value of a growing ordinary annuity (PVGA) is the sum of the present values of a series of periodic payments increasing at a constant percentage rate each year.

$$PVGA = R_1\ (PVIFGA_{i,n,g}) \qquad (4)$$

$$PVIFGA = \frac{1 - ([1 + g]/[1 + i])^n}{(i - g)} \qquad \text{for } i \neq g \qquad (5)$$

$$= n/(1 + i) \qquad \text{for } i = g \qquad (6)$$

where
PVIFGA = present value interest factor of a growing ordinary annuity;[2]

For example, to find the present value of a 3-year ordinary annuity that begins at $1,000 but increases at a 10% annual rate, discounted at 6%, see Figure 2.

FIGURE 2

TIME LINE FOR THE PRESENT VALUE OF A GROWING ORDINARY ANNUITY
(R_1 = $1,000; i = 6%; n = 3; and g = 10%)

End of Year

0 | 1 | 2 | 3

$1,000 $\qquad$ $(1 + .10)\$1,000$ $\qquad$ $(1 + .10)^2\$1,000$

Discounted 1 period
$943.40
Discounted 2 periods
978.89
Discounted 3 periods
1,015.94

$2,938.33 = Present Value of a Growing Ordinary Annuity(PVGA)

Or, according to Equations (4) and (5)

$$PVGA = R_1\ (PVIFGA_{6\%,3,10\%})$$

$$= (\$1,000)\ \frac{1 - ([1 + .10]/[1 + .06])^3}{(.06 - .10)}$$

$$= (\$1,000)\ (2.93833) = \underline{\$2,938.33}$$

TABLE 1

SPECIAL CASE: PVGA AND THE GORDON COMMON STOCK VALUATION MODEL

The well-known Gordon common stock valuation model states that:*

$$P_0 = \frac{D_1}{k_e - g} \quad (7)$$

where P_0 = current stock price (time 0);

D_1 = expected cash dividend at the end of one period;

k_e = market required return on the investment (or equity capitalization rate);

g = expected constant future dividend growth rate;

This is simply a special case of the present value of a growing ordinary annuity (PVGA):

$$PVGA = R_1 \left(PVIFGA_{k_e,\infty,g}\right)$$

$$P_0 = D_1 \frac{1 - ([1 + g]/[1 + k_e])^{\infty}}{(k_e - g)}$$

Assuming $k_e > g$, the term raised to an infinite exponent approaches zero, leaving

$$P_0 = D_1/k_e - g)$$

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* Equation (7) is often referred to in finance as the Gordon model for Myron Gordon, who pioneered its use. See Myron Gordon, The Investment, Financing, and Valuation of the Corporation, Homewood, Ill.: Irwin, 1962.

GROWING ANNUITY DUE CALCULATIONS

The FVIFGA and PVIFGA formulas are designed for ordinary (end-of-period) annuities. To convert these factors to an annuity due (beginning-of-period) basis, multiply the factor by (1 + i).

SAMPLE PROBLEMS

1. You plan to retire in twenty years. If the annual (year-end) amount you save each year increases annually at a 6 percent rate (the growth rate of your income) and will be $1,000 initially at year end, and if you can earn 8 percent on your savings, how much will your retirement fund be worth in 20 years?

   Answer: **$72,691**

2. The annual end-of-year lease payments on a building increase 10 percent annually for the next 5 years. If 8 percent is an appropriate interest rate, and if the first year's lease payment is $10,000, what is the most an investor would pay to be the recipient of these lease payments?

   Answer: **$48,043**

3. You wish to save annually an increasing (6% growth rate) amount (because your salary increases annually), and you wish to be worth $100,000 in 10 years. If you can earn 10% compounded annually on your savings, how much should your first (end-of-year) amount saved be?

   Answer: **$4,982**

4. Your Uncle Harvey expects to live another 10 years. (Should he live longer, he feels you would be pleased to provide for him.) He currently has $50,000 in savings which he wishes to spread evenly in terms of purchasing power over the remainder of his life. Since he feels inflation will average 6 percent annually, his annual **beginning-of-year** withdrawals should increase at a 6% growth rate. If he earns 8 percent on his savings not withdrawn, how much should his first withdrawal be? [Remember that the present value interest factor for a growing annuity due (PVIFGAD) is equal to the PVIFGA x (1 + i).]

   Answer: **$5,431**

FOOTNOTES

1. Taylor's [1] Appendix derives our Equation (2). Additionally, application of l'Hospital's rule to Equation (2), in the special case where i = g, yields our Equation (3).

2. Dividing Equation (2) by $(1 + i)^n$ yields Equation (5) -- i.e., turns a future value into a present value. The special case of i = g calls for the application of l'Hospital's rule to Equation (5), thus yielding Equation (6).

BIBLIOGRAPHY

1. Taylor, Richard W., "Future Value of a Growing Annuity: A Note." Journal of Financial Education (Fall 1986), 17-21.