Just Lindahl Taxation - A Welfarist Solution

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Abstract

The classic Lindahl allocation in a public good economy is both Pareto efficient and individually rational. However, it is easy to generate examples where the Lindahl outcome violates our intuition about economic justice. In this paper, we explore how a suitable generalization of Lindahl taxation can lead to fair outcomes. We generalize Lindahl's equilibrium approach so that consumers are given personalized price *schedules* for the public good (as opposed to simply personalized prices). The result is a special case of Mas-Colell and Silvestre's cost share equilibrium. We show that any outcome on the individually rational Pareto frontier can be achieved by some generalized Lindahl equilibrium. We then set up an optimization problem to search for a "just" Lindahl equilibrium. A social welfare function is first used to select an outcome on the individually rational Pareto frontier. We provide an algorithm to construct the price functions that induce the precise generalized Lindahl equilibrium that obtains this outcome. Finally, we present a mechanism that Nash implements the set of generalized Lindahl equilibria for our environment.

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1 Introduction

This paper is about fairness in the taxation of public goods - a topic with a long history in public finance. In 1896, Swedish economist Knut Wicksell first presented his unanimity principle of just taxation. Wicksell highlighted that the definition of justice in a public good setting is itself a problem and argued for a simple solution. To achieve economic justice, a society should unanimously choose both how public funds should be spent and how the burden of raising those funds should be shared. Since any proposal with unanimous support is a Pareto improvement, no individuals are harmed by undergoing the proposal. While Wicksell's argument might be compelling, it is not immediately clear how one goes about finding unanimity in a group of people.[1] This problem is first addressed by Lindahl (1919) who presented a market-like mechanism for making a public good allocation decision that achieves Wicksell's requirement of unanimity.

Lindahl's scheme for reaching a consensus goes as follows: In a constant marginal cost environment, Lindahl imagines a cost sharing mechanism which mimics a Walrasian auctioneer. First, the marginal cost of each unit produced is divided between the participants in the form of a price or cost share. Given these personalized prices, individuals respond with their demand for the public good. If all individuals demand the same level of the public good (i.e., if we have unanimity), the mechanism is in equilibrium and is stopped, the desired public good level is produced where individuals pay their individualized price for each unit of the public good produced. By construction, the tax revenue collected is equal to the cost of providing the public good.

The resulting allocation, hereafter Lindahl allocation, has a number of nice properties. It is efficient, individually rational, and shares a strong parallel with the standard Walrasian market equilibrium in the private good

[1] Wicksell's unanimity principle and the problem of reaching consensus is latent in the political economy work of James Buchanan and Gordan Tullock (1962). Incidentally, it is Buchanan who translated Wickell's article into English.

setting.[2] As a consequence, the Lindahl allocation has maintained relevance in the public finance literature often held as a *normative* benchmark for the basic public good setting. However, we are still left with the question...Is Lindahl taxation just? By design, of course, Lindahl's allocation achieves Wicksell's criterion for justice, namely unanimity. However, it is easy to generate examples (we provide one later in the paper) where the Lindahl outcome violates more modern notions of fairness and equity. This is not surprising since it was not Lindahl's intention to define a public good outcome that satisfies any society's definition of justice only Wicksell's. Nonetheless, we argue there is room for improvement.

In this paper, we explore how a suitable extension of Lindahl's idea can lead to efficient and fair public good outcomes. We define a generalized Lindahl equilibrium, hereafter GLE, where consumers are given personalized price schedules for the public good (as opposed to simply personalized prices). The extension to non-linear Lindahl prices is a more flexible tool for dividing social surplus than linear prices in the traditional set-up. This flexibility allows for more distributions of surplus to be attained in equilibrium. The classical Lindahl equilibrium is shown to be a special case of the GLE; and GLE are also shown to be special cases of Mas-Colell and Silvestre's cost-share equilibrium concept. Despite being a subset of the possible cost-share equilibria, we show that GLE are sufficiently flexible to achieve any outcome on the individually rational Pareto frontier, hereafter IRPF.[3] Specifically, given any outcome on the IRPF, we provide an algorithm for constructing the price functions that obtain the indicated outcome as a GLE. Next, we use a standard social welfare function to model a society's views on the equitable

[2] Lindahl equilibrium allocations have been shown to satisfy other nice properties as well. For example, van den Nouweland, Tijs, and Wooders (2002) provide axiomatic foundations for Ratio equilibrium (and therefore also to Lindahl Equilibrium in economies with constant returns to scale production). They establish that the ratio equilibrium is the unique solution that satisfies one person rationality, consistency, and converse consistency.

[3] This is the set of utility profiles that both Pareto efficient and where each individual's utility is at least as large as it was in the initial endowment with no public good production.

distribution of utility. Given this function we set up an optimization problem select a socially preferred outcome on the IRPF and then apply the aforementioned algorithm to solve for a "just" Lindahl equilibrium - i.e., a GLE that obtains the socially desired outcome on the IRPF. Finally, as in the classical concept, any GLE requires knowledge about each individual consumer's preferences in order to specify the appropriate individual price schedule. In fact, for some social welfare functions, the GLE concept may require even more knowledge about the individual preferences than the classic concept. While this is not an issue from a "normative" benchmark perspective, it would appear to be problematic from a practical perspective. Why would consumers reveal information about their preferences to the government if those preferences are going to determine their tax liability? One of the interesting features of the classic Lindahl concept was that, in some situations, it is possible to design a mechanism that induced a game whose non-cooperative equilibrium outcomes coincided with the Lindahl outcomes.[4] This is the so called implementation problem. We consider the analogous implementation problem for GLE and introduce a mechanism that Nash implements the set of GLE outcomes.

Literature Review

This paper contributes to a body of work concerned with the extension of Lindahl's original theory. The classical Lindahl equilibrium concept is formalized in Samuelson (1954, 1955) and Foley (1970). These approaches aligned the Lindahl concept with the Walrasian equilibrium in the standard general equilibrium model.[5] In addition, Foley showed that, like the Walrasian equilibrium, all Lindahl equilibria are in the core of a public good economy.[6] However, if the public good production function does not have

[4] This was first illustrated by mechanisms due to Hurwicz (1979) and Walker (1981).

[5] An extension of Lindahl pricing to economies with local public goods has been provided by Conley and Wooders (1998).

[6] In economies with local public goods Wooders (1997) establishes that asymptotically

constant returns to scale, there are problems. Kaneko (1977), for example, shows in economies with non-constant returns to scales, Lindahl equilibria may not be in the core or may not even exist. To overcome this feature, Kaneko introduces a ratio equilibrium, in which the total costs of production of public goods are distributed among consumers according to a vector of ratios. These ratios define taxes paid to firms such that the costs of provision are exactly covered and no firm earns profits. In the case of constant returns to scale production the ratio equilibrium outcome is identical to the classical Lindahl equilibrium outcome. As a result, ratio equilibrium in a constant returns to scale production environment inherits the same critique of the classic Lindahl concept that we offer in this paper. Mas-Colell and Silvestre (1989) develop the cost-share equilibrium which endogenously determines profit shares for a public firm, and shows that profits should be distributed according to the benefit an individual receives from a public good.[7] In our environment, we show the GLE concept is a special case of cost-share equilibrium. However, our restriction of cost-share equilibrium is without loss from a surplus division perspective. GLE are shown to be sufficiently flexible to achieve *any* outcome on the IRPF. Thus, there are no individually rational welfare gains missed by restricting attention to GLE. This result highlights a new attractive quality for cost-share equilibrium that to our knowledge has not been presented in the literature prior.

Our use of the social welfare function is similar to how it is used in optimal taxation theory. In optimal income taxation, society's preferences for fairness/income distribution tend to be captured with the use of a social welfare objective function. This idea goes back at least to the foundational work of Mirrlees (1971) and is seen in the literature as a flexible approach which

the core coincides with the equilibrium - all individuals of the same type in the same jurisdiction pay the same Lindahl prices.

[7] In a local public good economy, van den Nouweland and Wooders (2011) provide an extension of the ratio equilibrium and the cost share equilibrium called a share equilibrium for which they provide an axiomatization.

accommodates many different normative standards.[8] This is the perspective taken in our results. However, this is not the only approach. Saez and Stantcheva (2016), for example, use *marginal* social welfare weights when studying small, budget neutral, tax reforms and *locally* optimal taxation. We feel both approaches offer flexibility to the modeler and can shed light on the equitable distribution of utility in taxation models.

Finally, we contribute to the public good mechanism design literature. This is a large literature. Early examples are the mechanisms introduced by Groves and Ledyard (1977), Hurwicz (1979), and Walker (1981).[9] These mechanisms all induced games whose Nash equilibria allocations were efficient for the given public good economy. While Groves and Ledyard's mechanism is efficient, it does not always yield equilibrium outcomes that are individually rational - i.e., it does not implement Lindahl allocations. In contrast, the mechanisms of Hurwicz and Walker Nash implement the Lindahl allocations. Since Hurwicz and Walker, new mechanisms have been proposed which implement the classic Lindahl equilibrium allocations of a public good economy and induce games with other properties such as out-of-equilibrium feasibility, stability of equilibrium, or acceptability.[10] The mechanism we introduce here is similar in spirit to Van Essen (2013). Players in our mechanism submit "requests" for incremental amounts of the public good and a "guess" about the residual marginal cost needed to be covered by them at each unit. In equilibrium, the players' guesses are correct and the tax function reduces to a generalized Lindahl pricing function. Players can unilaterally set the level of the public good to anything they want. However, in

[8]See, for example, Lockwood and Weinzierl (2015), Weinzierl (2014), or Fleurbaey and Maniquet (2018).

[9]The dynamic procedures due to Dreze and Pousin (1971) and Malinvoud (1971) are also quite relevant. These procedures under truth telling, while not incentive compatible, yield efficient public good outcomes where consumers may pay a different price at each marginal unit. The procedures are parametric, but it is quickly seen that the set of possible outcomes in these MDP are all generalized Lindahl outcomes.

[10]See, for example, Chen (2002), Healy and Mathevet (2012), Van Essen (2013), or Van Essen and Walker (2017).

equilibrium, the only level of the public good that is optimal is the Pareto efficient level generated by the given generalized Lindahl pricing schedule. We show this mechanism Nash implements the set of GLE allocations.

2 The Public Good Economy

We consider a public good economy with $n \geq 2$ consumers. There is one private good, one public good, and a constant returns to scale production technology. We denote the quantity of the public good consumed by x, and the private good consumed by consumer i by y_i, where consumers are indexed by subscript i. Each consumer is characterized by the convex consumption set $C_i = \mathbb{R}_+ \times \mathbb{R}$, an initial endowment of the private good $\omega_i > 0$, and no initial endowment of the public good.

The public good is produced by a competitive firm using the private good as an input at a positive constant marginal cost of c – i.e., each unit of the public good x requires $c > 0$ units of the private good.

If $\Omega = \sum \omega_i$, then the largest amount of the public good that can be produced is $\frac{\Omega}{c}$. An allocation in this economy is an $(N+1)$- tuple $(x, y_1, ..., y_n) \in \mathbb{R}_+ \times \mathbb{R}^n$. Preferences over allocations, for each consumer i, are represented by a quasi-linear utility function

$$u_i\left(x, y_i\right) = \int_0^x v_i(m)dm + y_i,$$

where $\frac{\partial u_i}{\partial x} = v_i(x)$, where the valuation function $v_i(x)$ is i's marginal rate of substitution between y_i and x. We assume each v_i is continuously differentiable on $\left[0, \frac{\Omega}{c}\right]$ and $\frac{\partial^2 u_i}{\partial x^2} = v_i'(x) < 0$ for $x \in \left[0, \frac{\Omega}{c}\right]$.

We only consider economies with a unique and positive efficient level of the public good–that is, there is a unique $x^{PO} \in \left[0, \frac{\Omega}{c}\right]$ that satisfies the

Samuelson Marginal Condition

$$\sum_i v_i(x^{PO}) = c.$$

This is guaranteed by assuming $\sum_i v_i(0) > c$ and $\sum_i v_i(\frac{\Omega}{c}) < c$. We assume $v_i(x^{PO}) \geq 0$. This is sufficient to ensure that each individual i's marginal rate of substitution $v_i(x)$ is non-negative at each $x \in \left[0, x^{PO}\right]$. Finally, for each i, we suppose that the initial private endowments are sufficiently large: $\omega_i \geq v_i(0)\frac{\Omega}{c}$.

In this environment, a classical Lindahl equilibrium is a profile of individual prices (one price for each consumer) such that each consumer facing their individualized price demands the same level of the public good; and the total revenue collected from each consumer exactly finances the public good.

Definition 1: *The classical Lindahl equilibrium allocation of the public good economy is given by* $(x^L, y_1^L, ..., y_n^L)$ *and a* n*-tuple of Lindahl prices* $(p_1^L, ..., p_n^L) \in \mathbb{R}^n$ *such that (i)* $\sum_i p_i^L = c$; (ii) f*or each* i*, their equilibrium bundle* $\left(x^L, y_i^L\right)$ *satisfies*

$$x^L \in \arg\max_{x \in \mathbb{R}_+} u_i(x, \omega_i - p_i x^L),$$

where $y_i^L = \omega_i - p_i x^L$*.*

In the definition, part (i) is the zero profit or budget balance condition. A competitive firm producing the public good with a constant marginal cost c has a profit function equal to $\pi(x; p_1^L, ..., p_n^L) = \left(\sum_i p_i^L\right) x - cx$. The profit maximizing supply choice is thus one where $\sum_i p_i^L = c$. The firm earns zero profit. In the sequel we shall ignore the discussion of the firm and simply discuss budget balance to mean that the tax revenue collected is equal to the cost of producing the public good. Finally, it is well known that the Lindahl equilibrium allocation is $(x^L, y_1^L, ..., y_n^L)$ a Pareto optimal allocation.

3 Motivating Example: Injustice of Classical Lindahl Taxation

The classical Lindahl allocation is both Pareto optimal and individually rational. It is therefore "just" in the sense of Wicksell. However, it is easy to generate examples where the Lindahl allocation contrasts with more modern ideas of fairness or justice.

Suppose we have a small public good environment with two people, A and B whose preferences for the public good and their private good are described by the utility functions

$$u_A(x, y_A) = \int_0^x (20 - 2m)\, dm + y_A$$

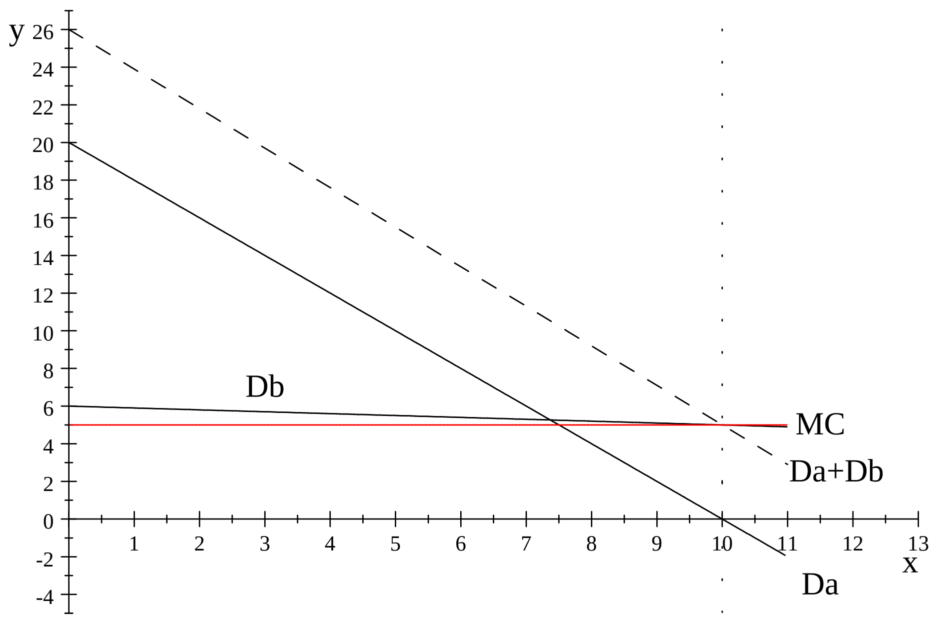
and

$$u_B(x, y_B) = \int_0^x \left(6 - \frac{1}{10}m\right) dm + y_B$$

respectively. In addition, we shall assume that each consumer is endowed with 100 units of the private good, none of the public good, and that the marginal cost of producing the public good is always 5. Thus, there is a unique Pareto optimal level of the public good at $x^{PO} = 10$ – the figure below illustrates.[11]

[11] This is found by solving the well known Samuelson Marginal Condition

$$MRS_A + MRS_B = MC.$$



Lindahl Equilibrium with (Potentially) Inequitable Distribution of Benefit

The classic Lindahl prices for Consumer A and B are $p_A^L = 0$ and $p_B^L = 5$ respectively. At these prices, both consumers demand the Pareto optimal level of the public good and the cost of producing the public good is exactly financed by the taxes collected. The allocation is efficient and the two consumers are better off relative to the status quo of no production. However, it would be quite a stretch to say that the allocation in the above example was "fair."

Consumer A gets a much better deal out of Lindahl Taxation than Consumer B. This is highlighted by comparing A's consumer surplus of

$$CS_A = \frac{1}{2}(20)(10) = 100$$

with B's consumer surplus of

$$CS_B = \frac{1}{2}(1)(10) = 5.$$

A pays zero in tax and receives significant benefit from the production of ten units of the public good. B, on the other hand, is only slightly better off than before production.

4 A Generalized Lindahl Equilibrium

A classical Lindahl equilibrium is *individually rational*, *Pareto optimal*, and *balances the budget* of providing of the public good. However, in the previous example, we saw how classical Lindahl prices might lead to an unjust division of the equilibrium surplus. Lindahl prices are linear and determined by evaluating each person's marginal benefit at the Pareto optimal level of the public good. By construction, these prices *ignore information* about an individual's marginal benefit of lesser units. In the example, A's marginal valuations for the first units are large relative to B's, but they diminish quickly and, at the PO level of the public good, the marginal valuation is exactly zero. Consumer B ends up paying the entire cost of the public good!

In this section, we generalize the classical Lindahl equilibrium concept by considering the benefits of using individualized price *schedules* instead of linear prices. A bundle of goods and a profile of price schedules will be called a generalized Lindahl equilibrium (GLE), if consumers, taking their personalized price schedules as given, demand the Pareto optimal level of the public good and if budget balance holds *per-unit* of the public good produced.

Definition 2: *A GLE of the public good economy is an allocation* $(x^{GL}, y_1^{GL}, ..., y_n^{GL})$ *and a n-tuple of personalized price schedules* $(p_1(\cdot), ..., p_n(\cdot))$ *such that (i)* $\sum_i p_i(m) = c$ *for each* $m \in [0, x^{GL}]$*, and (ii) for each* i*, their equilibrium bundle* $\left(x^{GL}, y_i^{GL}\right)$ *satisfies*

$$x^{GL} \in \arg\max_{x \in \mathbb{R}_+} u_i(x, \omega_i - \int_0^x p_i(m)dm),$$

where

$$y_i^{GL} = \omega_i - \int_0^{x^{GL}} p_i(m)dm.$$

In the definition of GLE, we have required personalized price schedules to satisfy *per-unit budget balance* (i.e., $\sum_i p_i(m) = c$ for each $m \leq x^{GL}$. This constraint is both and necessary and sufficient for the total cost of production to be covered by the aggregate tax revenue. This requirement is motivated by practical implementation considerations. The government's budget needs to be balanced in-equilibrium or out-of-equilibrium. Sufficiency of the requirement is clear. To see that per-unit budget balance is also necessary. Suppose we have a set of individual price functions that satisfy *aggregate* budget balance for all x, then the following equation holds

$$\int_0^x \sum_{i=1}^n p_i(m)dm = cx$$

for all x. Differentiating both sides gives us $\sum_{i=1}^n p_i(x) = c$ which shows that the price functions must also satisfy per-unit budget balance. Later, we show that this idea is consistent with Mas-Colell and Silverstre's notion of a cost-share system.

Proposition 1: *At any GLE* $(x^{GL}, y_1^{GL}, ..., y_n^{GL})$ *with* $(p_1(\cdot), ..., p_n(\cdot))$*, the equilibrium allocation is Pareto optimal.*

Proof: From the definition, $x^{GL} \in \arg\max_x u_i(x, \omega_i - \int_0^x p_i(m)dm)$. We first show $x^{GL} \neq 0$ is not part of a GLE. Therefore suppose $(0, y_1^{WL}, ..., y_n^{WL})$ is a GLE. Then a necessary condition for consumer i to demand $x = 0$ is

$$MRS_i = v_i(0) \leq p_i(0).$$

However, if we sum over all these conditions and apply (ii), then we get

$$\sum v_i(0) < c$$

This is a contradiction since the above violates the assumption $\sum_i v_i(0) > c$. Thus, $x^{GL} > 0$. The public good outcome therefore must satisfy the first order necessary condition

$$v_i(x^{GL}) = p_i(x^{GL})$$

for all i. If we aggregate this condition over all i, then

$$\begin{aligned}\sum_i v_i(x^{GL}) &= \sum_i p_i(x^{GL}) \\ &= c,\end{aligned}$$

where the equality follows from property (ii) of the definition. Since $v_i(x^{GL}) = MRS_i$, the above condition is simply the Samuelson Marginal Condition. Thus, $x^{GL} = x^{PO}$ and $p_i(x^{GL})$ is i's classic Lindahl price. Since each consumer's budget, by assumption, is sufficiently large, $y_i^{GL} = \omega_i - \int_0^{x^{GL}} p_i(m)dm \geq \omega_i - v_i(0)\frac{\omega}{c} > 0$. Finally, per-unit budget balance ensures that the cost of production is exactly covered. The GLE allocation is thus Pareto optimal. ■

Examples of GLE

GLE is a generalization of the classic Lindahl concept and contains the classic Lindahl solution as a special case. This is seen by defining each i's personalized price function as the constant function $p_i(t) = p_i^L$ for all $t \in [0, \frac{\Omega}{c}]$. The constant Lindahl price functions satisfy per-unit budget balance by construction of Lindahl prices. Furthermore, at this profile of personalized price functions, the demanded bundle of each consumer is simply their Lindahl equilibrium bundle. Thus, we have shown the following:

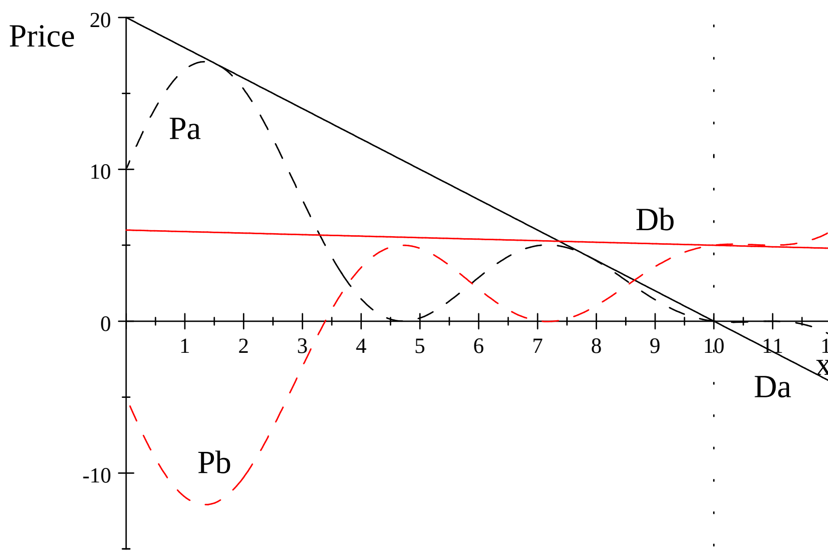
Proposition 2: *The classic Lindahl equilibrium is a GLE.*

Thus, anything (i.e., surplus/utility distribution) we achieve with a classic Lindahl equilibrium we achieve with a GLE. The point, however, is by using flexible personalized price schedules we can do more. The following examples illustrate.

There are typically many GLE. For instance, consider the public good environment in the motivating example where the Pareto optimal level of the public good is $x^{PO} = 10$. A GLE is induced with the personalized price schedules for A and B given by

$$p_A(x) = (10 - x)(\sin x + 1)$$
$$p_B(x) = 5 - (10 - x)(\sin x + 1)$$

respectively for each unit $x \geq 0$. This equilibrium is pictured below.



In this example, each consumer continues to demand 10 given their price schedule. Moreover, at each unit, the sum of the personalized prices is equal to 5 (the marginal cost). As a result, the total tax revenue collected is necessarily equal to the cost of production. These unusual price functions illustrate the flexibility of the GLE concept.

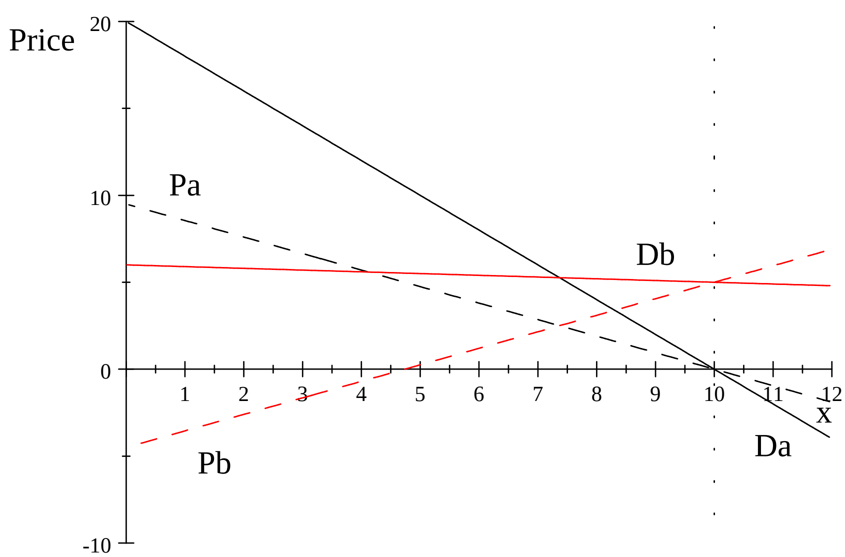
The flexibility of GLE can be used to achieve justice in taxation, as we discuss in more detail later in the paper where we will aim to construct price schedules that induce equitable equilibrium outcomes. As a preview, imagine we were in a society with Rawlsian preferences over the division of total surplus. In particular, suppose we offer consumer A the price schedule

$$p_A(x) = \frac{19}{2} - \frac{19}{20}x$$

and consumer B the price schedule

$$p_B(x) = -\frac{9}{2} + \frac{19}{20}x.$$

These prices functions are illustrated in the graph below.



At these price schedules, both people demand the Pareto optimal amount of the public good $x^{PO} = 10$. In addition, for each x, we see that the tax revenue collected is

$$\begin{aligned} p_A(x) + p_B(x) &= \left(\frac{19}{2} - \frac{19}{20}x\right) + \left(-\frac{9}{2} + \frac{19}{20}x\right) \\ &= 5. \end{aligned}$$

So the marginal cost is covered exactly for each unit. The resulting allocation is therefore individually rational and Pareto optimal. This is a GLE and, in contrast to the classic Lindahl equilibrium, there is less inequality in terms of consumer surplus. In particular, Consumer A's consumer surplus is

$$CS_A = \int_0^{10} \left(20 - 2m - \frac{19}{2} + \frac{19}{20}m\right) dm = \frac{105}{2}$$

and Consumer B's consumer surplus is

$$CS_B = \int_0^{10} \left(6 - \frac{1}{10}m + \frac{9}{2} - \frac{19}{20}m\right) dm = \frac{105}{2}.$$

The allocation is efficient and equitable to a society with Rawlsian preferences over surplus.

GLE are cost-share Equilibria

In this section, we establish that the GLE are a special case of Mas-Colell and Silvestre's cost-share equilibrium. We first need a pair of definitions.

Definition 3: *A cost-share system is a family of n functions $g_i : \mathbb{R}^n_+ \to \mathbb{R}$ such that $g_i(0) = 0$ and $\sum_i g_i(x) = cx$.*

Definition 4: *A cost-share equilibrium is a feasible allocation $(x^*, y_1^*, ..., y_n^*)$ and a cost-share system $(g_1, ..., g_n)$ such that (x^*, y_i^*) satisfies*

$$x^* \in \arg\max_x u_i(x, \omega_i - g_i(x)),$$

where $y_i^* = \omega_i - g_i(x)$ for each i.

A personalized price function p_i generates the personalized cost-share function $g_i(x) = \int_0^x p_i(m)dm$, where $g_i(0) = 0$. In a GLE, per-unit budget balance implies $\sum_i p_i(t) = c$ for $t \in [0, \frac{\Omega}{c}]$. As a consequence, we have the cost-share functions generated by GLE price functions satisfy

$$\begin{aligned}
\sum_i g_i(x) &= \sum_i \int_0^x p_i(m)dm \\
&= \int_0^x \sum_i p_i(m)dm \\
&= cx.
\end{aligned}$$

Thus, GLE price functions generate a valid cost-share system. Given this result and, by comparing Definition 4 with Definition 2, we have the following.

Proposition 3: *Let $(x^*, y_1^*, ..., y_n^*)$ and an n-tuple of personalized price schedules $(p_1^*(\cdot), ..., p_n^*(\cdot))$ be a GLE. The allocation $(x^*, y_1^*, ..., y_n^*)$ and the cost-share system $(g_1, ..., g_n)$ form a cost-share equilibrium, where the cost-share functions*

$$g_i(x) = \int_0^x p_i^*(m)dm$$

$i = 1, ..., n$.

GLE and the individually rational Pareto frontier

We have so far shown that GLE outcomes are Pareto optimal and a special case of the cost-share equilibrium concept. In this section, we show that *any* utility profile on the IRPF can obtained by a GLE. This result is important for two reasons: First, in our environment, it implies that GLE are sufficiently flexible tools for distributing social surplus. In particular, we are not missing potential individually rational welfare gains by restricting attention to GLE. Second, the proof of the result is constructive. It provides a step-by-step method for constructing a GLE to achieve any specified outcome on the IRPF. In the next section, we use this feature of the result combined with a social welfare function for selecting a point on IRPF in order to construct a "Just GLE" for a given society.

We begin by defining the Pareto frontier, the set of individually rational utility profiles, and the individually rational Pareto frontier (IRPF).

The *Pareto frontier* is the collection P of utility profiles $(u_1, ..., u_n)$ obtained from the possible Pareto efficient allocations. In our environment, since there is a unique Pareto optimal level of the public good and payoffs are quasi-linear, the Pareto frontier is given by

$$P = \{(u_1, ..., u_n) : \sum_{i=1}^{n} u_i = S\},$$

where

$$S \equiv \sum_{i=1}^{n} \mathring{y}_i + \sum_{i=1}^{n} \int_0^{x^{PO}} v_i(m)dm - cx^{PO}$$

is the total surplus (i.e., the value of the total endowment plus the social surplus created from the production of the Pareto optimal level of the public good). A utility profile $(u_1, ..., u_n)$ is *individually rational* if the utility achieved by individual i is at least as great as the utility they could achieve with zero public good production – i.e., $u_i \geq \mathring{u}_i \equiv \mathring{y}_i$. The the set of individually rational utility profiles is denoted I. Finally, the *individually rational Pareto frontier* is the set $I \cap P$.[12]

Proposition 4: *Suppose* $(u_1, ..., u_n) \in I \cap P$*, then there exists a GLE whose equilibrium allocation achieves the utility profile* $(u_1, ..., u_n)$*.*

Proof: We first show that it is possible to construct personalized price functions that raise the requisite amount of tax revenue from each consumer and satisfy the per-unit budget balance and per-unit individual rationality. We subsequently establish that the constructed personalized price functions are indeed a GLE.

[12] Since the utility profile corresponding to the Lindahl equilibrium outcome belongs to this set. We observe $I \cap PO$ is always non-empty.

For a given profile $u = (u_1, ..., u_n) \in I \cap P$, we have

$$u_i = \mathring{y}_i - t_i + \int_0^{x^{PO}} v_i(m)dm,$$

where $u_i \geq \mathring{y}_i$. Individual i's implied total tax is

$$t_i = \mathring{y}_i - u_i + \int_0^{x^{PO}} v_i(m)dm.$$

Since the allocation is Pareto optimal the total tax revenue is equal to the total cost of production – i.e.,

$$\sum_{i=1}^{n} t_i = cx^{PO}.$$

Denote i's Lindahl tax by $t_i^L = p_i^L x^{PO}$.

Next, we sort the individuals into two groups: A and B. The individuals whose taxes are "above" their Lindahl tax (i.e., $t_i \geq t_i^L$) are assigned to A. Individuals whose taxes are "below" their Lindahl tax (i.e., $t_i < t_i^L$) are assigned to B.

This partition of individuals yields the following tax identity. If we aggregate the total taxes paid by the members of A and B, then we have

$$\sum_{i \in A} t_i \geq \sum_{i \in A} t_i^L$$

and

$$\sum_{j \in B} t_j < \sum_{j \in B} t_j^L.$$

Since both

$$\sum_{i \in A} t_i^L + \sum_{j \in B} t_j^L = cx^{PO} \quad \text{and} \quad \sum_{i \in A} t_i + \sum_{j \in B} t_j = cx^{PO},$$

we have

$$\sum_{i\in A}\left(t_i - t_i^L\right) = \sum_{j\in B}(t_j^L - t_j).$$

In words, the excess tax paid by members of A above their Lindahl tax exactly offsets the amount in the tax paid by members of B below their Lindahl tax. This is intuitive. Decreases in taxes for one group have to be made up by the other group to cover the total cost of production.

Next, we construct individualized personalized prices starting with individuals in set A.

For $i \in A$, we have that $t_i \geq t_i^L$. Consider a personalized price function of the form

$$p_i^A(m) = \theta_i\left(v_i(m) - p_i^L\right) + p_i^L$$

for $m \leq x^{PO}$, $\theta_i \in [0,1]$, and equal to p_i^L otherwise. For $m \leq x^{PO}$, this price function selects a convex combination of i's marginal valuation function $v_i(t)$ and i's Lindahl price. Individual i's total tax is then equal to

$$T_i^A(\theta_i) = \theta_i \int_0^{x^{PO}} \left(v_i(m) - p_i^L\right) dm + p_i^L x^{PO}.$$

If $\theta_i = 1$, then this individual is paying their complete marginal valuation for the public good at each unit. This corresponds to the highest possible tax compatible with individual rationality and we denote this tax by $\bar{t}_i$. If $\theta_i = 0$, then this individual is paying their Lindahl price for each unit of the public good at each unit. The existence of $\theta_i \in [0,1]$ such that $T_i(\theta_i) = t_i$ then follows from the continuity of T_i in θ_i and the Intermediate Value Theorem (since $t_i^L \leq t_i \leq \bar{t}_i$). Given $T_i(\theta_i) = t_i$, we solve directly for θ_i. This determines $p_i^A(m)$ for $i \in A$.

Next, we construct individualized personalized prices for individuals in B.

First, we observe that no $j \in B$ has $t_j^L - t_j > \sum_{i\in A}\left(t_i - t_i^L\right)$. If this were so, then since $t_j < t_j^L$ for all $j \in B$, we would have

$$\sum_{j \in B} (t_j^L - t_j) > \sum_{i \in A} \left(t_i - t_i^L\right),$$

which would contradiction the result that

$$\sum_{i \in A} \left(t_i - t_i^L\right) = \sum_{j \in B} (t_j^L - t_j).$$

Thus, for each $j \in B$, there is a share $\delta_j \in [0,1]$ such that $\delta_j \sum_{i \in A} \left(t_i - t_i^L\right) = t_j^L - t_j$. Adding up across members of B yields

$$\sum_{j \in B} \left(t_j^L - t_j\right) = \left(\sum_{i \in A} \left(t_i - t_i^L\right)\right) \left(\sum_{j \in B} \delta_j\right) = \sum_{i \in A} \left(t_i - t_i^L\right).$$

Hence, $\sum_{j \in B} \delta_j = 1$.

We have identified the shares $(\delta_j)_{j \in B}$. Next, let j's personalized price function be defined by

$$p_j^B(m) = p_j^L - \delta_j \left[\sum_{i \in A} \theta_i \left(v_i(m) - p_i^L\right)\right]$$

for $m \leq x^{PO}$ and p_j^L otherwise. In words, j's personalized price is their Lindahl price minus their share δ_j of the marginal excess-Lindahl tax revenue from group A. If we add up the tax revenue for j we have

$$\int_0^{x^{PO}} p_j(m)dm = t_j^L - \delta_j \sum_{i \in A} \left(t_i^A - t_i^L\right) = t_j$$

as desired.

The constructed personalized price functions satisfy both per-unit individual rationality and per-unit budget balance. Per-unit individual rationality is satisfied by construction. Members of A face a per-unit price that is always between their marginal valuation function $v_i(m)$ and their Lindahl price. Members of B face a per-unit price that is always less than their Lindahl price (and therefore less than their marginal valuation function). Next, we show the total tax collected at unit m is equal to the marginal cost of production. In particular,

$$\sum_{i\in A} p_i^A(m) + \sum_{j\in B} p_j^B(m)$$

$$=$$

$$\sum_{i\in A} \left[\theta_i \left(v_i(m) - p_i^L\right) + p_i^L\right] + \sum_{j\in B} \left[p_j^L - \delta_j \sum_{i\in A} \theta_i \left(v_i(m) - p_i^L\right)\right]$$

$$=$$

$$\sum_{i\in A} \theta_i \left(v_i(m) - p_i^L\right) - \sum_{i\in A} \theta_i \left(v_i(m) - p_i^L\right) + \sum_{i\in A} p_i^L + \sum_{j\in B} p_j^L$$

$$=$$

$$\sum_{i\in A} p_i^L + \sum_{j\in B} p_j^L$$

$$=$$

$$c,$$

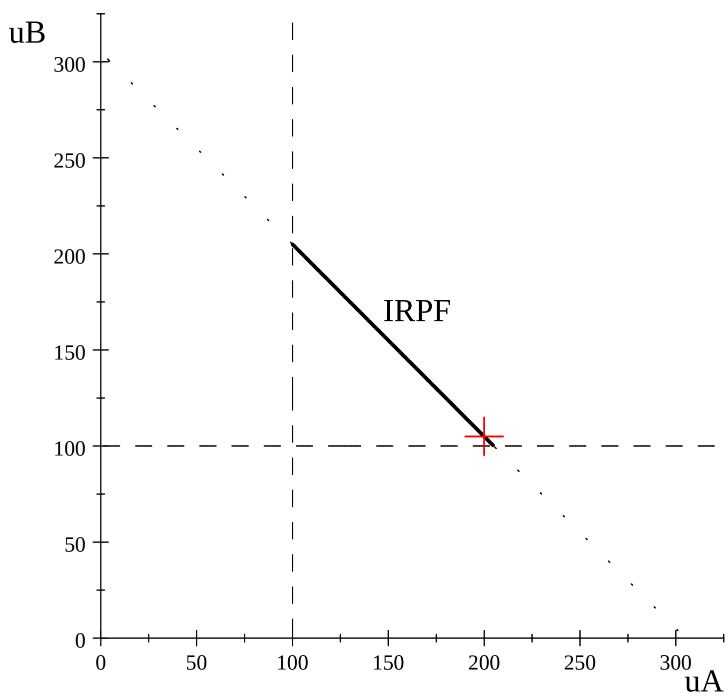
where the second equality follows from the fact that $\sum_{j\in B} \delta_j = 1$, and the third equality follows since the sum of all the Lindahl prices equals the marginal cost. We can conclude that per-unit budget balance is satisfied.

Finally, we verify that these personalized price functions induce a GLE. From per-unit individual rationality, each individual will always demand a public good level of at least x^{PO}. However, since each i's price function becomes their Lindahl price at x^{PO}, no individual demands more than x^{PO}. Thus, all individuals demand the same level of the public good, x^{PO}. The remainder of the definition is satisfied from per-unit budget balance. ■

If we return to our motivating example, the IRPF is

$$I \cap P = \{(u_A, u_B) : u_A + u_B = 305, u_A \geq 100, u_B \geq 100\}.$$

This set is illustrated by the thick line labeled *IRPF* in the graph below. Also illustrated are the dashed lines representing the individual rationality constraints. The proposition establishes that any point on the frontier can be obtained by a GLE. In contrast, there is only one utility profile achievable for the Lindahl/ ratio equilibrium in this example $(u_A, u_B) = (200, 105)$, illustrated by the small "+" in the figure.



Individually Rational Pareto Frontier from Motivating Example

Discussion: Constructing a Just Lindahl Equilibrium

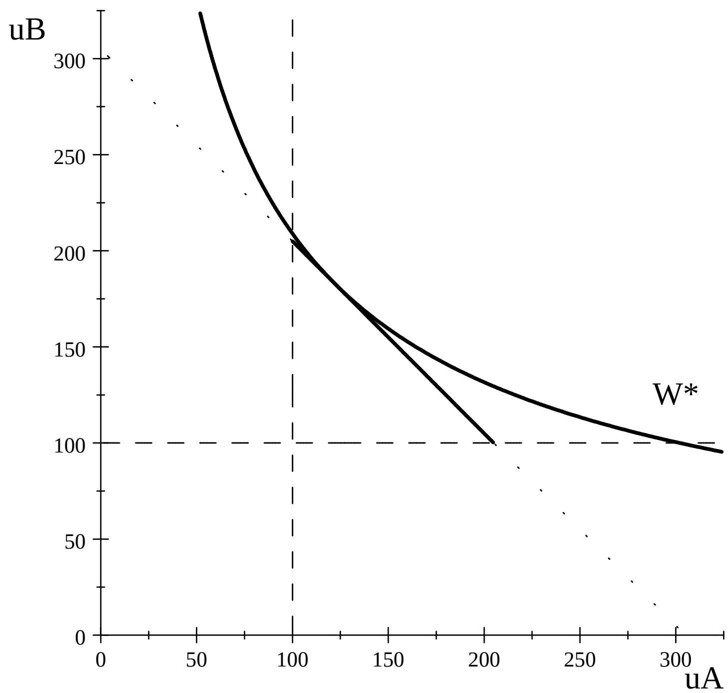
For a given public good environment, we have shown in the previous section that any point on the IRPF may be achieved by a GLE. However, not all of these points need be socially desirable. As seen in our motivating classic Lindahl equilibrium example, some GLE outcomes may not have any socially desirable properties other than efficiency and mild individual rationality. Thus, the question is, "Which outcome in the IRPF should we choose to implement?"

To address this problem, we introduce *social welfare function* W to encode society's views on the equitable distribution of the consumer utility. In particular, let $W : \mathbb{R}^n \to \mathbb{R}$ be any continuous function. This function takes as input a utility profile $(u_1, ..., u_n) \in I \cap P$ and returns a number $W(u_1, ..., u_n)$. Since the set $I \cap P$ is seen to be non-empty, convex, and compact, by the Weierstrass Theorem, the equitable distribution of utility problem

$$\max_{(u_1,...,u_n)\in I\cap P} W(u_1, ..., u_n)$$

has a well defined solution. Next, given such a solution $(u_1, ..., u_n)$, we can find a "Just" GLE that implements it using the algorithm introduced in the proof of Proposition 4.

We illustrate the construction of a Just GLE for the motivating example when the social welfare function is Cobb-Douglas of the form $W(u_A, u_B) = u_A^2 u_B^3$.



Equitable Distribution of Utility when Social Welfare Function is
$W(u_A, u_B) = u_A^2 u_B^3$

In the example, the equitable distribution solution is $(u_A, u_B) = (122, 183)$. This is illustrated in the figure where W^* is the graph of the level curve of W through the solution.

We now construct a GLE that achieves this particular outcome. First, individual A and B's implied taxes at this utility profile are $t_A = 78$ and $t_B = -28$ (i.e., a subsidy). We observe individual A's tax is above their Lindahl tax $t_A^L = 0$ and B's tax is below their Lindahl tax $t_B^L = 50$. Hence, from the proof of Proposition 4, A's personalized price is of the form

$$
\begin{aligned}
p_A(m) &= \theta_A (20 - 2m) + (1 - \theta_A)(0) \\
&= \theta_A (20 - 2m) .
\end{aligned}
$$

The share θ_A that achieves the requisite tax revenue is $\theta_A = \frac{78}{100}$.[13] Therefore, A's personalized price function is given by

$$p_A(m) = \begin{cases} \frac{78}{100}(20 - 2m) & \text{- if } m \leq 10 \\ 0 & \text{- otherwise.} \end{cases}$$

B claims $\delta = 1$ of the A's surplus tax revenue $t_A - t_A^L = 78$. The resulting price function is

$$p_B(m) = \begin{cases} 5 - \frac{78}{100}(20 - 2m) & \text{-if } m \leq 10 \\ 5 & \text{-otherwise} \end{cases}.$$

These personalized price functions induce a GLE which achieves the solution to the specified equitable distribution of utility problem.

5 Nash Implementation of GLE

In this section, we discuss GLE from the perspective of mechanism design. We provide a mechanism that Nash implements the set of GLE. In particular, all Nash equilibrium outcomes are shown to be GLE outcomes and, for each GLE, we can find a Nash equilibrium of the game that achieves this outcome as its equilibrium allocation.

A mechanism is a game form. It defines what messages players are allowed to send and an outcome function that maps players' messages into an outcome. Here we consider a mechanism in which players report messages to

[13] This is since we need

$$\begin{aligned} 78 &= \theta_A \int_0^{10} (20 - 2m)\, dm = 100\theta_A \\ &\rightarrow \\ \theta_A &= \frac{78}{100}. \end{aligned}$$

a "planner" who uses this information to determine an outcome consisting of an amount of the public good x to produce and a tax τ^i for each player.

In the Residual Cost Covering Mechanism, each player i submits a message $\mathbf{m}_i = (r_i, w_i(\cdot)) \in \mathbb{R} \times \mathbb{C}[0, \frac{\Omega}{c}]$ to the planner, where the number r_i represents their incremental request for the public good and the continuously differentiable function $w_i(\cdot)$ on $[0, \frac{\Omega}{c}]$. The set $\mathbb{C}[0, \frac{\Omega}{c}]$ denotes the set of continuously differentiable functions on $[0, \frac{\Omega}{c}]$. The amount $w_i(t)$ specifies the most they are willing to pay in order to get an additional amount of the public good at any possible unit $t \in [0, \frac{\Omega}{c}]$. We denote the profile of messages by $\mathbf{m} = (\mathbf{m}_1, ..., \mathbf{m}_n)$. The messages are collected by the planner and used to determine the level of the public good x and a tax τ^i for each player i according to outcome functions $\chi(\mathbf{m})$ and $\tau^i(\mathbf{m})$ respectively, where

$$\chi(\mathbf{m}) = \begin{cases} 0 & \text{- if } \sum_i r_i < 0 \\ \sum_i r_i & \text{- if } 0 \leq \sum_i r_i \leq \frac{\Omega}{c} \\ \frac{\Omega}{c} & \text{- otherwise.} \end{cases}$$

$$\tau^i(\mathbf{m}) = \begin{cases} \infty & \text{- if } \sum_i r_i \notin [0, \frac{\Omega}{c}] \\ \int_0^{\chi(\mathbf{m})} c - \sum_{j \neq i} w_j(z) dz + \max\limits_{t \in \left[0, \frac{\Omega}{c}\right]} \left| \sum_{k=1}^n w_k(t) - c \right| & \text{- otherwise.} \end{cases}$$

The next proposition establishes the implementation result.

Proposition 5: *The Residual Cost Covering mechanism Nash implements the set of GLE.*

Proof: A mechanism with outcome functions $\chi(\mathbf{m})$ and $\tau^i(\mathbf{m})$ induces a normal form game between players where, for each player i, the messages get mapped into payoffs according to $u_i(\chi(\mathbf{m}), \omega - \tau^i(\mathbf{m}))$.

Suppose the message profile $(r_i^*, w_i^*(\cdot))_{i=1}^n$ forms a Nash equilibrium of the game induced by the Residual Cost Covering mechanism. Let x^* be the public good produced and τ_i^* be the tax i pays in this equilibrium. We need to show that the equilibrium outcome is a GLE outcome.

A profile $(r_i^*, w_i^*(\cdot))_{i=1}^n$ such that $\sum_i r_i^* \notin [0, \frac{\Omega}{c}]$ cannot be part of Nash equilibrium. A player can always select $r_i^* = -\sum_{j\neq i} r_j^*$ and $w_i^*(t) = c - \sum_{j\neq i} w_j^*(t)$ at each t and achieve a zero payoff. Hence, $x^* = \sum_i r_i^* \in [0, \frac{\Omega}{c}]$.

Given the messages of players $j \neq i$, in any best response player i chooses $w_i^*(t) = c - \sum_{j\neq i} w_j^*(t)$ at each t. Otherwise i pays unnecessary additional tax.

Next, given the request announcements of the other players, each player i can unilaterally set the public good level $x = \chi(\mathbf{m})$ to be any level they want. The three possible cases are that $x = 0$, $x \in (0, \frac{\Omega}{c})$, or $x = \frac{\Omega}{c}$.

If $x^* = 0$, then the best response request r_i^* must satisfy

$$\begin{aligned} v_i(0) &\leq c - \sum_{j\neq i} w_j^*(r_i^* + \sum_{j\neq i} r_j^*) \\ &= w_i(0). \end{aligned}$$

However, for this to be a Nash profile, the above condition needs to by true for each i. Summing across all i we have $\sum_i v_i(0) \leq c$ which contradicts the assumption that $\sum_i v_i(0) > c$. Hence, $x^* > 0$.

If $x^* = \frac{\Omega}{c}$, then the best response request r_i^* must satisfy

$$\begin{aligned} v_i(\frac{\Omega}{c}) &\geq c - \sum_{j\neq i} w_j^*(r_i^* + \sum_{j\neq i} r_j^*) \\ &= w_i\left(\frac{\Omega}{c}\right). \end{aligned}$$

However, for this to be a Nash profile, the above condition needs to by true for each i. Summing across all i we have $\sum_i v_i(\frac{\Omega}{c}) \geq c$ which contradicts the assumption that $\sum_i v_i(\frac{\Omega}{c}) < c$. Hence, $x^* \in (0, \frac{\Omega}{c})$.

The best response request r_i^* must therefore satisfy the interior first order condition

$$\begin{aligned} MRS^i &= c - \sum_{j \neq i} w_j^*(r_i^* + \sum_{j \neq i} r_j^*) \\ &= c - \sum_{j \neq i} w_j^*\left(x^*\right) \\ &= w_i\left(x^*\right) \end{aligned}$$

If we sum this condition across all i we recover the Samuelson Marginal Condition

$$\sum_i MRS^i = \sum_i w_i\left(x^*\right) = c.$$

Thus, the x^* is Pareto optimal. The outcome is budget balanced at each unit since $\sum_i w_i^*\left(t\right) = c$ for each $t \in [0, x^*]$ and, thus, the total amount of taxes collected from the players is exactly equal to the cost of producing the public good. The outcome must be individually rational since players can achieve a zero payoff by choosing $w_i^*\left(t\right) = c - \sum_{j \neq i} w_j^*(t)$ at each t and $r_i = -\sum_{j \neq i} r_j^*$. Finally, since each individual is demanding the same level of the public good and their total tax liability equals the cost of producing the public good, the equilibrium outcome is the GLE induced by the personalized tax function w_i^* for each i.

Now, pick a GLE outcome with public good $\tilde{x}$ and a tax $\tilde{\tau}^i$ for each player i. We need to find a Nash equilibrium outcome of the game induced by the Residual Cost Covering Mechanism that achieves this GLE outcome.

Since we have a GLE outcome, the total tax revenue collected from i for z units is $\tilde{\tau}^i(z) = \int_0^z p_i(t)dt$ for some personalized price function $p_i(t)$. Set $r_i = \frac{\tilde{x}}{n}$ and $w_i\left(t\right) = p_i(t)$ for each t and each i. We need to show that these messages form a Nash equilibrium and that the equilibrium outcome recovers the desired GLE outcome.

First, by definition of GLE, $\sum_i w_i\left(t\right) = \sum_i p_i(t) = c$ for each t. Hence, $w_i\left(t\right) = c - \sum_{j \neq i} w_j(t)$ for each t and i's willingness-to-pay function announcement is a best response to the rivals' willingness-to-pay function announcements. Next, given the willingness-to-pay announcements of their rival's and i's own willingness-to-pay announcement, i's optimal request determines the public good level x which is the solution to

$$\max_x u_i(x, \int_0^x c - \sum_{j \neq i} w_j(t)dt)$$

where $x = r_i + \sum_{j \neq i} r_j$. However, since $w_i(t) = p_i(t)$ for each t this is exactly the demand problem of the GLE. Hence, the solution is $x = \tilde{x}$ and therefore $r_i = \frac{\tilde{x}}{n}$ is optimal for i. In summary, the message profile where $r_i = \frac{\tilde{x}}{n}$ and $w_i(t) = p_i(t)$ for each t and each i is a Nash equilibrium and the equilibrium outcome coincides with the specified GLE – that is, we have $x(\mathbf{m}) = \tilde{x}$ and $\tau^i(\mathbf{m}) = \int_0^{\tilde{x}} \left(c - \sum_{j \neq i} w_j(z)\right) dz = \int_0^{\tilde{x}} p_i(z)dz = \tilde{\tau}^i$ for each i. ■

6 Discussion

The classic Lindahl equilibrium is a cornerstone of the equitable taxation literature. Like the Walrasian equilibrium allocation, the Lindahl allocation is Pareto optimal and individually rational and therefore represents a potential normative benchmark for a public good economy. However, since the classic concept only uses constant personalized prices as its vehicle to generate unanimity we have shown examples where this can lead to unfair outcomes. In the paper, we extended the Lindahlian idea to one personalized tax schedules. A GLE is a set of personalized price schedules such that all individuals demand the same level of the public good and whose tax revenue exactly covers the cost of production. In our environment, this equilibrium concept is a special case of Mas-Colell and Silvestre's cost-share equilibrium. The use of non-linear personalized prices to generate unanimity is more flexible and allows us to be more discriminatory in the assignment of consumer surplus.

Our main result shows that GLE can achieve and point on the IRPF. This result and the associated algorithm allow us to find GLE to implement any socially desired outcome consistent with efficiency and individual rationality. The last section of the paper discusses the implementation problem where we provide an example of a mechanism that Nash implements the GLE outcomes in the public good economy.

Finally, we have made strong assumptions on the environment primitives– a constant returns-to-scale production technology, quasi-linear utility functions with nice derivatives, large initial endowments, etc. These restrictions produced an environment with a unique Pareto optimal level of the public good, let us easily compute the IRPF, and provided a clean setting with no income effects and transferable utility to study the GLE implementation problem. However, the simple environment does not push the envelope on what is possible. In addition, the constant returns to scale environment hides important differences between Lindahl equilibrium and other concepts (e.g. the ratio equilibrium or the cost-share equilibrium). The logical extension of our ideas to environments to finding desirable cost-share equilibria in environments with non-linear production technology seems like a natural next step. Other areas of pursuit for future research would be to consider zero cost public goods, externalities and Pigouvian taxation, or utility functions with income effects.

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