Equity Equilibrium for Cooperative Games

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Abstract

We introduce a unifying stability concept to cooperative game theory–the equity equilibrium. A central authority selects an outcome of the game to enforce and evaluates its stability using a collection of functions called a "complaint system." These complaints are used to identify the grievances against and the concessions to each player. Equity equilibrium occurs when an individually rational payoff configuration balances the grievances and concessions of each player. We establish the existence of equity equilibrium for any valid complaint system and under any coalition structure. Next, we show that equity equilibrium under specific complaint systems characterizes the kernel, the Shapley value, and the generalized Nash bargaining solution of a cooperative game. We show how simplicial algorithms can be employed for computing any type of equity equilibrium. This approach is illustrated with an example from the Tennessee Valley Authority.

Key Words: Cooperative Games, Equity Equilibrium, Kernel, Shapley Value, Computation of Cooperative Solutions
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Only on the haughty rich,
And on their unjust store,
He'd lay his fines of equity
For his merry men and the poor.

– from "How Robin and His Outlaws Lived in The Woods," by Leigh Hunt (1820).

1 Introduction

In cooperative games with transferable utility, players arrange themselves into groups and then decide how to divide their coalition's surplus. Solution concepts for these games focus on outcomes that can be achieved through *stable* agreements between players. A variety of solutions have been proposed and analyzed in the literature, each with their own motivation and adherents. In this paper, we introduce a new and unifying concept of stability–the equity equilibrium.

A game, in our equilibrium framework, is administered by a central authority that is responsible for selecting the game's outcome. The center evaluates the stability of all possible outcomes by anticipating the players' objections. Formally, this forecast is given by a collection of real-valued functions called a complaint system. This system specifies, for each outcome, the (anticipated) complaint of each player against every other player in the game. The center uses these complaints to identify, for each player, the "total grievance" against as well as the "total concession" to that player. Grievances signal that the player's payoff is too large; whereas concessions signal that the player's payoff is too small. Outcomes where the complaints against a player are too one-sided are considered inequitable and require equity adjustments. The authority's goal is to choose an equilibrium outcome where no equity adjustments are needed, given the forecasted objections.

We establish the existence of equity equilibrium for *any* coalition structure under general conditions.[1] For each player, we define an excess concession function which, for each outcome, is the total amount conceded to the player net the total grievance against them. This function is continuous since the complaint system is assumed to be continuous. Next, we establish a complementarity identity akin to Walras' Law in general equilibrium theory. Proposition 1 is our basic existence result. The proof of the proposition uses well-known fixed point methods.[2] We define an appropriate outcome transition function and then apply Brouwer's fixed point theorem to establish that this transition function has a fixed point. Next, the complementarity identity is used to show that the fixed point is indeed an equity equilibrium. The assumptions for the basic existence of equilibrium are minimal. However, not all complaint systems lead to interesting equity equilibria. In general, we find that the complaint systems based on the primitives of games tend to have more structure. We give two "characterization lemmas" that provide basic properties about equity equilibria for "well-behaved" complaint systems. These lemmas are used in all of our later characterizations of classic solutions.

Equity equilibrium is next shown to characterize several classic solutions from cooperative game theory under specific complaint systems: the kernel, the Shapley value, and the generalized (i.e., weighted) Nash bargaining solution.[3] Our equivalence results are found in Propositions 2, 3, and 4 for the kernel, Shapley value, and generalized Nash bargaining solution, respectively. These different characterizations are useful for several reasons. First, they

[1]For our existence result, we impose two requirements on the complaint system. First, the complaint system needs to be continuous. Second, players' complaints about players not in their coalition are set to zero. Complaint systems with these two features are called "valid."

[2]For example, see chapter 2 in Scarf (1973).

[3]The kernel was introduced in Maschler and Davis (1965); the Shapley value in Shapley (1953); and the original Nash bargaining solution in Nash (1950).

illustrate how disparate solution concepts can be linked under a common equilibrium framework. This provides a *cohesive* interpretation for each of these solutions. In addition, since the classic solutions are well-studied (and linked), their characterizations relate equity equilibrium to other related solution concepts (the pre-kernel, bargaining set, core, nucleolus, etc.). These concepts are often motivated by stories of bargaining,[4] where equity equilibrium provides an alternative arbitration-type interpretation that might be a better fit for some applications. Second, the common characterization facilitates *comparison* of solutions. We can now directly compare these different solution concepts by comparing their complaint systems. Third, if a general algorithm for computing equity equilibria can be found,[5] then this algorithm can be used to *compute* any other cooperative solution characterized by equity equilibrium. Finally, these characterizations suggest *corollary* concepts. For example, the Shapley value, while defined for the grand coalition, can be extended to outcomes where more than one coalition is formed. We call such solutions "Shapley value consistent" and provide an equity equilibrium characterization for these outcomes. The analogous extension and the equity equilibrium characterization are also made for the generalized Nash bargaining solution.

Finally, to be applicable, we need a reliable way of computing an equity equilibrium. While our existence proof utilizes Brouwer's fixed point theorem and is non-constructive, we can utilize simplicial algorithms (a lá Scarf) to approximate equity equilibria. This approach provides a unified compu-

[4]The bargaining foundations of these cooperative solutions have been explored in the *non-cooperative literature*. Rubinstein (1982) is connected to the Nash bargaining solution in Binmore (1987). Perry and Reny (1994) study a bargaining model related to the core. Harsanyi (1977), Gul (1989), and Hart and Mas-Colell (1996) are examples that provide non-cooperative foundations for the Shapley value. Finally, Serrano (1997) provides a bargaining foundation for the (pre) kernel.

[5]We provide such an algorithm at the end of the paper. Computation is discussed later in the introduction.

tational method for finding equity equilibria. We provide a brief "reader friendly" survey of simplicial algorithms and discuss to what extent these algorithms compute approximate economic equilibria (including equity equilibria). Next, we illustrate how an *off-the-shelf* simplicial algorithm can be used to approximate an equity equilibrium. In particular, we provide a step-by-step method for the computation of an equity equilibrium, based on Kuhn's artificial start algorithm (Kuhn 1968), in a game where players form a grand coalition. While other algorithms have been proposed in the literature, Kuhn's method is perhaps the easiest one to describe and, for the neophyte, the simplest to program.[6] Finally, we illustrate this approach by using Kuhn's algorithm to compute both the kernel and the Shapley value for a famous example involving the Tennessee Valley Authority.[7]

Literature Review

We make several contributions to the literature.

Cooperative Game Theory Solutions:

This paper proposes an equilibrium concept for cooperative games. Given a complaint system, any associated equity equilibrium identifies a stable outcome. By judiciously choosing the complaint system, as we show later, the equity equilibrium can characterize classic solutions such as the kernel, the Shapley value, and the generalized/weighted Nash bargaining solution (as well as partially characterize the epsilon core). Thus, the flexibility of the complaint system allows us to use a common language to compare and contrast different cooperative solutions. Naturally, each classic solution has its

[6]Kuhn's algorithm is used for computing equity equilibria when players form the grand coalition. For equity equilibria involving multiple non-singular coalitions, we refer the reader to the simplicial algorithms detailed in Doup (1988).

[7]See, for example, Young (1994) or Straffin and Heany (1980).

own literature and has connections with other solutions such as the core, the bargaining set, Walrasian and Lindahl equilibria, the pre-kernel, and the nucleolus.[8] Thus, by varying the complaint system, the equity equilibrium is shown to be related to a myriad of other classic concepts. Peleg and Sudhölter (2007) is the main reference for cooperative game theory solutions and many of their interconnections.

Computation of Economic Equilibria and Cooperative Solutions:

The question of whether equity equilibria can be computed is a valid one. Equity equilibria are shown to exist, but our proof uses Brouwer's fixed point theorem and is therefore non-constructive. Fortunately, there is a large literature on using algorithms that approximate such fixed points.[9] Any one of these algorithms allows us to compute any cooperative solution concept that is characterized by equity equilibria. In the paper, we present a simplicial algorithm to compute approximate equity equilibria, based on Kuhn (1968). However, in the literature, there are many other simplicial algorithms that are theoretically more efficient.[10]

[8]The bargaining set was introduced in Aumann and Maschler (1964). The pre-kernel is introduced in Maschler, Peleg, and Shapley (1979). The nucleolus by Schmeidler (1969). Two good surveys of the bargaining set, the kernel, and the nucelolus can be found in Maschler (1992) and Iñarra, Serrano, and Shimomura (2019). The recent paper of Gul and Pesendorfer (2024) relates the weighted Nash bargaining solution in assignment problems without transfers with the Lindahl equilibrium.

[9]This literature begins with Scarf (1967a,b). A history of the early literature and the ideas behind these algorithms can be found in either Scarf (1973) or Scarf (1982). Books by Doup (1988) and Yang (1999) offer a variety of different fixed point algorithms, extensions of the basic algorithms to the simplotope, and applications.

[10]Homotopy methods described in Yang (1999) are good examples. The "sandwich method" described in Kuhn and MacKinnon (1975) or MacKinnon (1974) is a particular homotopy algorithm that uses much of the same machinery as Kuhn (1968). Thus, the algorithm presented in this paper for approximating equity equilibrium can be converted to the sandwich method if more efficiency is desired.

The alternative to simplicial methods is a specific algorithm for each solution concept. The advantage of this approach is that one may take advantage of the special structure of a specific concept and, hopefully, generate a more accurate and efficient algorithm. However, it is unknown whether these algorithms are more efficient than simplicial methods. Algorithms for finding solutions in the bargaining set, the kernel, the pre-kernel, and the Shapley value have been proposed by several authors.[11] Aumann, Peleg, and Rabinowitz (1965) provide an early algorithm for computing the kernel; Stearns (1968) provides an iterative scheme that converges in the limit to the kernel (and another for the bargaining set); Harsanyi (1977) provides an algorithm for computing the Shapley value, using coalitional dividends. Harsanyi's coalitional dividends play an important role in our equity equilibrium characterization of the Shapley value. Different algorithms based on using linear programming methods at each step are known for computing the kernel, the pre-kernel, and the nucleolus. These methods are discussed in Peleg and Sudhölter (2007). Chalkiadakis, Elkind, and Wooldridge (2012) provides a survey of computational methods for finding different cooperative solutions. Finally, Meinhardt (2014) discusses recent advances in algorithms for the computation of the pre-kernel.

2 Cooperative Games and Equity Equilibrium

Cooperative Games and their Outcomes

A *cooperative game* Γ is a pair (N, v), where $N = \{1, ..., n\}$ is the set of players and v is the characteristic function that maps each coalition $S \subseteq N$ into payoff $v(S)$, where $v(S) \geq 0$ for all $S \subseteq N$ and $v(\{i\}) = 0$ for all $i \in N \cup \{\emptyset\}$. An outcome in (N, v) is defined by the coalitions that are formed

[11] The pre-kernel elements and the kernel elements coincide when the game is superadditive and players form the grand coalition.

by the players and the distribution of each coalition's payoff among their group. An *individually rational payoff configuration* (*IRPC)* is an outcome $(\mathbf{x};\boldsymbol{\beta})$, where $\mathbf{x}$ is a vector of shares, one for each $i \in N$, and $\boldsymbol{\beta} = \{\beta_1, ..., \beta_m\}$ is a partition of N into $m \geq 1$ disjoint coalition structures. The share vector $\mathbf{x}$ in an IRPC is such that $x_i \geq 0$ for all i and, for each coalition β_j, we have $\sum_{i \in \beta_j} x_i = 1$.[12] Hence, the IRPCs are the outcomes in which each coalition β_j's shares belong to the $(|\beta_j| - 1)$-dimensional unit simplex, denoted X^{β_j}. Finally, the Cartesian product $X \equiv \times_{i=1}^{m} X^{\beta_i}$.

Game Administrator

The game (N, v) is administered by a central authority, hereafter the center, which selects an IRPC $(\mathbf{x};\boldsymbol{\beta})$ and forecasts the stability of the outcome using a collection of functions called a "complaint system." A "satisfactory" solution for the center is an IRPC that is "stable" under the given complaint system (i.e., an equity equilibrium). We now work to make these ideas precise.

Complaint Systems, Stability, and Equity Equilibrium

A *complaint system* $\mathbb{C}$ is defined by an $n \times n$ matrix of functions, where each function maps an outcome $(\mathbf{x};\boldsymbol{\beta})$ into a real number. The (k, l) element of $\mathbb{C}$, denoted $c_{k,l}(\mathbf{x};\boldsymbol{\beta})$, is interpreted by the center as player k's (anticipated) complaint against player l at $(\mathbf{x};\boldsymbol{\beta})$. A complaint system $\mathbb{C}$ is *valid* under $\boldsymbol{\beta}$, if for each $k, l \in N$ we have that $c_{k,l}(\mathbf{x};\boldsymbol{\beta})$ is a continuous function in $\mathbf{x}$; and if k and l are not in the same coalition under $\boldsymbol{\beta}$, then $c_{k,l}(\mathbf{x};\boldsymbol{\beta}) \equiv 0$ for all $\mathbf{x}$.

Relative complaints between pairs signal to the center whose payoff share may need an equity adjustment. We divide relative complaints into two categories: "grievances against" and "concessions to" each player. A *grievance*

[12] Given $(\mathbf{x};\boldsymbol{\beta})$, if $i \in \beta_j$, then i's payoff in Γ is $v(\beta_j)x_i$ – that is, i receives an x_i share of coalition j's surplus.

against player k occurs when another player l's complaint about k is larger than k's complaint about l. We quantify l's grievance against k under $(\mathbf{x}, \boldsymbol{\beta})$ as $x_l\left(c_{l,k}\left(\mathbf{x}; \boldsymbol{\beta}\right) - c_{k,l}\left(\mathbf{x}; \boldsymbol{\beta}\right)\right)$ – that is, a weighted share of the relative complaint. Analogously, concessions occur when a player l's complaint about player k is smaller than k's complaint about l. We quantify l's concession to k under $(\mathbf{x}, \boldsymbol{\beta})$ as $x_l\left(c_{k,l}\left(\mathbf{x}; \boldsymbol{\beta}\right) - c_{l,k}\left(\mathbf{x}; \boldsymbol{\beta}\right)\right)$. These quantities are aggregated. The total grievance against k, denoted TG_k, is the sum of all of the individual players' grievances against k. Likewise, the total concession to k, denoted TC_k, is the sum all the individual players' concessions to k.

Example 1 illustrates the computation of grievances and concessions for a given complaint system.

Example 1: Consider the complaint system

$$\mathbb{C} = \begin{bmatrix} c_{AA} & c_{AB} & c_{AC} \\ c_{BA} & c_{BB} & c_{BC} \\ c_{CA} & c_{CB} & c_{CC} \end{bmatrix} = \begin{bmatrix} 0 & 3x_B - 1 & 3x_C - 2 \\ 3x_A - 3 & 0 & 3x_C - 2 \\ 3x_A - 3 & 3x_B - 1 & 0 \end{bmatrix}$$

Suppose the coalition is the grand coalition and shares $\mathbf{x} = (x_A, x_B, x_C) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ are proposed, then this outcome is an IRPC. The grievances against and concessions to player A are

	B	C	Total
Grievance vs. A:	0	0	0
Concession to A:	$\frac{2}{3}$	$\frac{1}{3}$	1

Hence, $TC_A > TG_A$ at $\mathbf{x}$. The grievances against and concessions to player B are

	A	C	Total
Grievance vs. B:	$\frac{2}{3}$	$\frac{1}{3}$	1
Concession to B:	0	0	0

Hence, $TC_B < TG_B$ at $\mathbf{x}$. Finally, the grievances against and concessions to player C are

	A	B	Total
Grievance vs. C:	$\frac{1}{3}$	0	$\frac{1}{3}$
Concession to C:	0	$\frac{1}{3}$	$\frac{1}{3}$

Hence, $TC_C = TG_C$.

$\Diamond$

From the center's perspective, grievances are a signal that a player's payoff share may be too high and concessions are a signal that a player's payoff may be too low. In Example 1, $TC_A > TG_A$ suggests that A's payoff share is too low, $TC_B < TG_B$ suggests that B's payoff share is too high, whereas $TC_C = TG_C$ suggests that C's payoff share is fine. Outcomes where these signals offset each other are considered stable. Specifically, player k's share x_k is *stable* at $(\mathbf{x}, \boldsymbol{\beta})$ under $\mathbb{C}$ if either

1. $TG_k = TC_k$; or

2. $TG_k > TC_k$ and $x_k = 0$.

In Example 1, only player C's payoff share is stable.

An IRPC $(\mathbf{x}, \boldsymbol{\beta})$ is in *equity equilibrium* under $\mathbb{C}$ if each player's share is stable.

Example 2: Returning to the complaint system in Example 1. If

$$\mathbf{x} = (x_A, x_B, x_C) = (\frac{2}{3}, 0, \frac{1}{3}),$$

we have that $TG_i = TC_i = 0$ for $i = A$, B, and C. Hence, each player's payoff share is stable at $\mathbf{x}$. The IRPC $(\mathbf{x}, \boldsymbol{\beta})$ forms an equity equilibrium for the grand coalition $\{A, B, C\}$ under the complaint system in Example 1.$\Diamond$

In Example 2, we found an equity equilibrium without knowing the game. In general, interesting equity equilibria are generated using complaint systems derived from the primitives of an underlying cooperative game. In the next example, as we show later in the paper, the illustrated equity equilibrium coincides with the game's Shapley value.

Example 3: Consider the super-additive game[13] (N, v), where $N = \{A, B, C\}$ and the characteristic function is

$$\begin{aligned} v(\{A,B,C\}) &= 3, \\ v(\{A,B\}) &= 1,\ v(\{A,C\}) = v(\{B,C\}) = 2, \\ v(\{A\}) &= v(\{B\}) = v(\{C\}) = 0. \end{aligned}$$

The shares $x_A = x_B = \frac{5}{18}$ and $x_C = \frac{8}{18}$ form an equity equilibrium for the grand coalition under the complaint system

$$\mathbb{C} = \begin{bmatrix} c_{AA} & c_{AB} & c_{AC} \\ c_{BA} & c_{BB} & c_{BC} \\ c_{CA} & c_{CB} & c_{CC} \end{bmatrix} = \begin{bmatrix} 0 & 3x_B - \frac{3}{2} & 3x_C - 2 \\ 3x_A - \frac{3}{2} & 0 & 3x_C - 2 \\ 3x_A - \frac{3}{2} & 3x_B - \frac{3}{2} & 0 \end{bmatrix}.$$

This follows since the outcome is an IRPC and at $\mathbf{x}$ we have $c_{AB} = c_{BA}$, $c_{AC} = c_{CA}$, and $c_{BC} = c_{CB}$. Thus, $TG_i = TC_i = 0$ for $i = A, B, C$. $\Diamond$

The complaint systems used in the examples are typical and of the following form: If $k, l \in \beta_j$, then $c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) = v(\beta_j)x_l - b_l$ with b_l a constant; and $c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) = 0$ otherwise. This family of complaint systems is valid and turns out to be useful in several of our characterizations. As such, it is useful to classify them. Hereafter we refer to a member of this family of complaint systems as a *threshold complaint system*.

[13] Recall that cooperative game (N, v) is *super additive* if for any $S_1, S_2 \subseteq N$ we have $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$. A super-additive game provide incentives for the players to form the grand coalition $\{N\}$.

3 Existence of Equity Equilibrium and its Properties

In this section, we establish the existence of equity equilibrium points for coalitional games with valid complaint systems and include some properties of equilibrium.

Existence of Equity Equilibrium

Fix a partition of the players into coalitions $\boldsymbol{\beta}$ as well as any valid complaint system $\mathbb{C}$, then for each player $k \in N$, we define the mapping $g_k : X \to \mathbb{R}$ where

$$\begin{aligned} g_k(\mathbf{x}) &= x_1 \left(c_{k,1}(\mathbf{x}) - c_{1,k}(\mathbf{x})\right) + \\ &\quad x_2 \left(c_{k,2}(\mathbf{x}) - c_{2,k}(\mathbf{x})\right) + \\ &\quad \ldots + \\ &\quad x_n \left(c_{k,n}(\mathbf{x}) - c_{n,k}(\mathbf{x})\right). \\ &= TC_k - TG_k. \end{aligned}$$

This is the “excess concession” function for player k. Under this notation, an IRPC $(\mathbf{x}, \boldsymbol{\beta})$ is in equity equilibrium if, for each k, we have either $x_k \geq 0$ and $g_k(\mathbf{x}) = 0$; or $g_k(\mathbf{x}) < 0$ and $x_k = 0$.

Lemma 1: *If $\mathbb{C}$ is a valid complaint system, then, for each $k \in N$,* the mapping *g_k is a continuous function in x.*

Next, we establish a complementarity identity that is analogous to Walras' Law from general equilibrium theory.

Lemma 2: *Suppose the complaint system $\mathbb{C}$ is valid, then for all $(\mathbf{x}, \boldsymbol{\beta})$ and each coalition $\beta_j \in \boldsymbol{\beta}$ we have*

$$\sum_{k \in \beta_j} x_k g_k(\mathbf{x}) = 0.$$

The existence of equilibrium follows from the continuity of the excess concession functions, the above complementarity identity, and an application of Brouwer's fixed point theorem.

Proposition 1: *For any coalition structure β and any associated valid complaint system $\mathbb{C}$ there exists an equity equilibrium.*

Some Properties of Equity Equilibria

Given a complaint system $\mathbb{C}$, we define the *out-complaints relation* $\rhd$ on N as follows: $k \rhd l$ if and only if $c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) > c_{l,k}(\mathbf{x}, \boldsymbol{\beta})$ and $x_l > 0$. In many complaint systems of interest, as we show later, $\rhd$ is a strong partial order on N (i.e., it is irreflexive, asymmetric, and transitive). The equity equilibria associated with such complaint systems have predictable structures. The following lemma reports some useful properties of an equity equilibrium associated with $\mathbb{C}$ whose induced relation $\rhd$ is transitive. These results are used in several of our characterizations.

Lemma 3: *Suppose the relation $\rhd$ induced by $\mathbb{C}$ is transitive. If $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium associated with $\mathbb{C}$, then the following statements are true:*
(i) If $c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) > c_{l,k}(\mathbf{x}, \boldsymbol{\beta})$, then $x_l = 0$.
(ii) For any pair k, $l \in N$ we have

$$(c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) - c_{l,k}(\mathbf{x}, \boldsymbol{\beta}))x_l \leq 0$$

and

$$(c_{l,k}(\mathbf{x}, \boldsymbol{\beta}) - c_{k,l}(\mathbf{x}, \boldsymbol{\beta}))x_k \leq 0.$$

(iii) For any pair k, $l \in N$ such that $x_k > 0$ and $x_l > 0$ we have

$$c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) = c_{l,k}(\mathbf{x}, \boldsymbol{\beta}).$$

Threshold complaint systems, as in Example 3, always induce an out-complaints relation $\rhd$ that is a strong partial order on N. The special structure of this complaint system provides us with some additional characterization results, which are presented in Lemma 4.

Lemma 4: *Suppose $\mathbb{C}(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b})$ is a threshold complaint system with constants $\mathbf{b} = (b_1, ..., b_n)$. Then the following statements are true:*
(i) The out-complaints relation $\rhd$ induced by $\mathbb{C}$ is transitive.
(ii) If there exists an $\bar{\mathbf{x}} \in X$ such that $(\bar{\mathbf{x}}, \boldsymbol{\beta})$ is an IRPC and, for each coalition β_j, we have

$$c_{k,l}(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b}) = c_{l,k}(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b})$$

for all $l,k \in \beta_j$; then every equity equilibrium $(\mathbf{x}, \boldsymbol{\beta})$ under $\mathbb{C}$ for the partition $\boldsymbol{\beta}$ has

$$c_{k,l}(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b}) = c_{l,k}(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b}) = c_{k,l}(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b})$$

for all $l,k \in \beta_j$. In addition, if $v(\beta_j) > 0$ for coalition j, then $\bar{\mathbf{x}}$ identifies the unique payoff shares for members of coalition j in every equity equilibrium under partition $\boldsymbol{\beta}$.

4 Characterizations of Classical Solutions using Equity Equilibrium

Proposition 1 shows that equity equilibria exist under reasonable conditions. However, not all complaint systems generate interesting equity equilibria (e.g., the constant complaint system).[14] In this section, we show that known cooperative solutions can be characterized as equity equilibria.

The kernel and Equity Equilibrium

[14] There is probably a joke in there somewhere!

The kernel is one of the basic solutions in cooperative game theory. In this section, we show that the kernel is characterized by the equity equilibrium concept under a specific complaint system.

Review of the Kernel

The kernel of a cooperative game is a pair-wise stability concept à la Maschler and Davis (1965). It's definition requires two concepts: the "excess" of a coalition and the "surplus" one player has over another.

Let $(\mathbf{x};\boldsymbol{\beta})$ be an IRPC. The *excess* of coalition D with respect to $(\mathbf{x};\boldsymbol{\beta})$, denoted $e_D(\mathbf{x},\boldsymbol{\beta})$, is defined by

$$e_D(\mathbf{x},\boldsymbol{\beta}) \equiv v(D) - \sum_{j=1}^{m}\left[\sum_{i\in D\cap\beta_j} v(\beta_j)x_i\right].$$

The excess of D represents the total amount that the members D (could) gain if they withdraw from $(\mathbf{x};\boldsymbol{\beta})$ and formed coalition D. For player $k \in \beta_j$, if $k \in D$, then $e_D(\mathbf{x},\boldsymbol{\beta})$ represents a potential bargaining chip k could use against another member, $l \in \beta_j$, when arguing for a higher share of $v\left(\beta_j\right)$. Of course, if $l \in \beta_j$ and $l \in D$, then player l could use the same $e_D(\mathbf{x},\boldsymbol{\beta})$ against k. This motivates the definition of one player's surplus over another player.

If two distinct players k and l are in the same coalition, then we define player k's *surplus* over l, denoted $s_{k,l}(\mathbf{x},\boldsymbol{\beta})$, to be the maximum excess player k could gain by withdrawing from $(\mathbf{x};\boldsymbol{\beta})$ and joining a coalition that does not also contain player l. The surplus is defined to be zero in all other cases. Thus, k's surplus over l is given by

$$s_{k,l}(\mathbf{x},\boldsymbol{\beta}) = \begin{cases} \max_{D\in T_{k,l}} e_D(\mathbf{x},\boldsymbol{\beta}) & \text{if } k \neq l \text{ are in the same coalition,} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$T_{(k,l)} = \{D | D \subseteq N,\ k \in D, \text{ and } l \notin D\}.$$

Let $(\mathbf{x}, \boldsymbol{\beta})$ be an IRPC for Γ, and let k, l be two distinct players in a coalition β_j. Player k is said to "outweigh" player l with respect to $(\mathbf{x}, \boldsymbol{\beta})$, denoted $k \gg l$, if

$$s_{k,l} > s_{l,k} \quad \text{and} \quad x_l \neq 0.$$

If neither $k \gg l$ nor $l \gg k$, then we say that k and l are balanced and we denote it by $k \approx l$. The *kernel* K is the set of IRPCs $(\mathbf{x}, \boldsymbol{\beta})$ such that every pair of players $k, l \in N$ is balanced.

Remark: Maschler and Davis (1965, *Lemma 5.1*) established that the outweigh relation $\gg$ is a strict partial order.

Equity Equilibrium Characterization of the Kernel

The surplus function of Maschler and Davis defines a valid complaint system. The *Maschler-Davis Complaint System*, denoted $\mathbb{C}^{MD}(\mathbf{x}, \boldsymbol{\beta})$, is defined by the $n \times n$ matrix whose (k, l)-th element $c_{k,l}^{MD}(\mathbf{x}, \boldsymbol{\beta}) \equiv s_{k,l}(\mathbf{x}, \boldsymbol{\beta})$. By inspection, this complaint system is valid and induces an out-complaint relation $\rhd$, which is identical to the outweigh relation $\gg$. Lemma 5 follows immediately from these observations.

Lemma 5: *The following statements about $\mathbb{C}^{MD}$ are true:*
(i) $\mathbb{C}^{MD}$ is a valid complaint system.
(ii) The out-complaints relation $\rhd$ induced by $\mathbb{C}^{MD}$ is a strict partial order.

Proposition 2 shows that the kernel elements of the coalitional game coincide with the equity equilibrium outcomes under $\mathbb{C}^{MD}$.

Proposition 2: *The IRPC $(\mathbf{x}, \boldsymbol{\beta}) \in K$ if and only if $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium under $\mathbb{C}^{MD}$.*

This equivalence of equity equilibrium and the kernel under $\mathbb{C}^{MD}$ has some immediate corollaries. From Lemma 5, $\mathbb{C}^{MD}$ is valid, we therefore know from Proposition 1 that the set of equity equilibria under $\mathbb{C}^{MD}$ is non-empty and, therefore, from Proposition 2, K is non-empty. It is known (from Maschler and Davis) that if $(\mathbf{x}, \boldsymbol{\beta}) \in K$, then $(\mathbf{x}, \boldsymbol{\beta})$ is in the bargaining set. Hence, the non-emptiness of the bargaining set also follows from Proposition 1 and Proposition 2.

Corollary 1: *The kernel and the bargaining set are both non-empty.*

If the game is super additive and the specified coalition is the grand coalition, then the kernel is known to be the same as the pre-kernel.

Corollary 2: *An IRPC* $(\mathbf{x}, \boldsymbol{\beta})$ *for a super-additive game is in the pre-kernel for the grand coalition if and only if it is an equity equilibrium under* $\mathbb{C}^{MD}$ *for the grand coalition.*

Example 4: Consider the super-additive game (N, v), from Example 3 under the Maschler-Davis complaint system $\mathbb{C}^{MD} =$

$$\begin{bmatrix} 0 & \max\{-3x_A, 2-3x_A-3x_C\} & \max\{-3x_A, 1-3x_A-3x_C\} \\ \max\{-3x_B, 2-3x_B-3x_C\} & 0 & \max\{-3x_B, 1-3x_B-3x_C\} \\ \max\{-3x_C, 2-3x_B-3x_C\} & \max\{-3x_C, 2-3x_A-3x_C\} & 0 \end{bmatrix}.$$

The grand coalition with payoff shares $\mathbf{x} = \left(\frac{2}{9}, \frac{2}{9}, \frac{5}{9}\right)$ is an equity equilibrium under $\mathbb{C}^{MD}$. From Proposition 2, we know this outcome is in the kernel of the game (N, v) from Example 3. From Corollaries 2 and 3 we have that this outcome is in the bargaining set and is in the pre-kernel. $\Diamond$

THE SHAPLEY VALUE AND EQUITY EQUILIBRIUM

In this section, we construct a different complaint system that characterizes the Shapley value as an equity equilibrium. In fact, the characterization does more. It includes the Shapley value consistent outcomes, which are defined on proper cooperative subgames of the original game.[15]

[15] We define these items later in this section.

Review of the Shapley Value

The Shapley value is the unique solution for dividing the grand coalition's surplus that satisfies the following axioms: (i) the grand coalition's surplus is fully divided; (ii) symmetric players receive identical payoffs; (iii) dummy players receive a zero payoff; and (iv) additivity. In addition, if the game (N, v) is super additive, then its Shapley value is also efficient and individually rational.

Following Harsanyi (1977), the Shapley value can be computed using "coalitional dividends" as follows: Given a cooperative game (N, v), for each coalition $R \subseteq N$, there is a unique constant c_R associated with it. This constant is determined inductively by setting $c_\emptyset \equiv 0$ and then, for all other $R \subseteq N$, $c_R \equiv v(R) - \sum_{Q \subset R} c_Q$.[16] Then, for each non-empty coalition $R \subseteq N$, the *coalitional dividend* for R, denoted w_R, is defined as the constant c_R divided by the number players in the coalition–that is, $w_R \equiv \frac{c_R}{|R|}$. The Shapley value payoff $\phi_i(v)$ for each player $i \in N$ is then given by[17]

$$\phi_i(v) = \sum_{R \subseteq N, i \in R} w_R.$$

Example 5: *Suppose* $N = \{A, B, C\}$ *with characteristic function:*

$$\begin{aligned} v(\{A,B,C\}) &= 3, \\ v(\{A,B\}) &= 1,\ v(\{A,C\}) = v(\{B,C\}) = 2, \\ v(\{A\}) &= v(\{B\}) = v(\{C\}) = 0. \end{aligned}$$

[16] We use "$\subset$" to denote a *strict* subset–that is, $A \subset B$ is used for subsets of B that are not equal to B. In contrast, we use "$\subseteq$" to denote a weak subset–that is, $A \subseteq B$ if either $A \subset B$ or $A = B$.

[17] Equivalently, for a general characteristic function v, the Shapley value ϕ_i of player i is

$$\phi_i = \sum_{S \subseteq \{1,\ldots,N\}} \frac{(|S|-1)!(N-|S|)!}{N!} \left[v(S) - v(S \backslash \{i\}) \right].$$

The coalitional dividends for each coalition are

$$
\begin{array}{lll}
w_{\{A\}} = 0 & w_{\{B\}} = 0 & w_{\{C\}} = 0 \\
w_{\{A,B\}} = \frac{1}{2} & w_{\{A,C\}} = 1 & w_{\{B,C\}} = 1 \\
 & w_{\{A,B,C\}} = -\frac{2}{3} &
\end{array}
$$

The Shapley value shares are given by $\sum_{R \subseteq N, i \in R} \frac{w_R}{v(\{A,B,C\})}$ for each i, or $x_A = x_B = \frac{5}{18}$ *and* $x_C = \frac{8}{18}$. ◊

Cooperative Subgames and Shapley Value Consistent Outcomes

In some games, players have no incentive to form the grand coalition. However, Shapley's idea can be extended to these coalitional games by looking at the cooperative subgames induced by $\boldsymbol{\beta}$. Given a coalition S of players, the *cooperative subgame* induced by S is the game defined by the player set S and characteristic function $v|_S$, which is the restriction of the original characteristic function v to coalitions only involving the players in S. A coalition structure $\boldsymbol{\beta}$ is a partition of the players into coalitions. If, for each $S \in \boldsymbol{\beta}$, the cooperative subgame induced by S is super additive, then we say that the coalition structure $\boldsymbol{\beta}$ satisfies *subgame super additivity.*

Since a cooperative subgame is a game, all solution concepts that apply to cooperative games continue to hold for cooperative subgames. Given a cooperative subgame $(S, v|_S)$, we can compute the Shapley value for that subgame and it can be computed independently of other subgames by using the same coalitional dividends we would use to compute the Shapley value for the original game. We say that an IRPC $(\mathbf{x}, \boldsymbol{\beta})$ is *Shapley consistent*, if $\mathbf{x}$ prescribes the Shapley value division of surplus for each cooperative subgame induced by the coalitions in $\boldsymbol{\beta}$.

Example 6: Consider the following "Decreasing Returns" cooperative game. Suppose $N = \{A, B, C, D\}$, where the value of a one player coalition is 0, the value of a two player coalition is 3, the value of a three player coalitions is 2, and the value of the four player coalitions is 1. This game is not super additive. However, players may want to form two-player coalitions. For example, the cooperative subgame induced by coalition $\{A, B\}$ is the one with player set $\{A, B\}$ and the following restricted characteristic function

$$\begin{aligned} v|_{\{A,B\}}(\{A, B\}) &= 3 \\ v|_{\{A,B\}}(\{A\}) &= 0 \\ v|_{\{A,B\}}(\{B\}) &= 0 \end{aligned}$$

Suppose players form coalitions $\boldsymbol{\beta} = \{\{A, B\}, \{C, D\}\}$. This coalition structure satisfies subgame super additivity. The payoff shares $\mathbf{x}$ corresponding to the unique Shapley value consistent outcome for $\boldsymbol{\beta}$ are

$$\mathbf{x} = (x_A, x_B, x_C, x_D) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}).$$

Since the two cooperative subgames induced by $\boldsymbol{\beta}$ are super-additive, the Shapley consistent outcomes are efficient (within their subgame) and individually rational.$\diamondsuit$

Equity Equilibrium Characterization of Shapley Value Consistent Outcomes

In this section, we use equity equilibrium to characterize the Shapley value consistent outcomes of the game. The complaint system is defined using Harsanyi's coalitional dividends.

The *Shapley Complaint System*, denoted $\mathbb{C}^S(\mathbf{x}, \boldsymbol{\beta})$, is the $n \times n$ matrix whose (k, l)-th element is defined as follows:

$$c^S_{k,l}(\mathbf{x}, \boldsymbol{\beta}) = \begin{cases} v(\beta_j)x_l - \sum_{l \in R, R \subset \beta_j} w_R & \text{- if } k \neq l \text{ are in the same coalition, and} \\ 0 & \text{- otherwise.} \end{cases}$$

Proposition 3: *Suppose (N, v) is subgame super additive under $\boldsymbol{\beta}$, then $(\mathbf{x}, \boldsymbol{\beta})$ is Shapley value consistent for $\boldsymbol{\beta}$ if and only if $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium under $\mathbb{C}^S$.*

If $\boldsymbol{\beta}$ is the grand coalition, then Proposition 3 connects equity equilibrium with the standard Shapley value.

Corollary 3: *Suppose* (N, v) *is super additive, then* $(\mathbf{x}, \{N\})$ *is the Shapley value outcome if and only if* $(\mathbf{x}, \{N\})$ *is an equity equilibrium under* $\mathbb{C}^S$*.*

The reader may check that the complaint system used in Example 3 is a Shapley complaint system (using the coalitional dividends obtained in Example 6) and that the equity equilibrium shares for Example 3 coincide with the Shapley value of the game (also given in Example 6).

A sufficient condition for the game to be super additive is for it to be convex.[18] When a cooperative game is convex, the core is non-empty and contains the Shapley value.[19] Thus, when the game is convex, the equity equilibria for the grand coalition under $\mathbb{C}^S$ are contained in the core.

Example 7: The Shapley Complaint System for the Decreasing Returns Game (Example 6) for $\boldsymbol{\beta} = \{\{A, B\}, \{C, D\}\}$ is

$$\mathbb{C}^S = \begin{bmatrix} c_{AA} & c_{AB} & c_{AC} & c_{AD} \\ c_{BA} & c_{BB} & c_{BC} & c_{BD} \\ c_{CA} & c_{CB} & c_{CC} & c_{CD} \\ c_{DA} & c_{DB} & c_{DC} & c_{DD} \end{bmatrix} = \begin{bmatrix} 0 & 3x_B & 0 & 0 \\ 3x_A & 0 & 0 & 0 \\ 0 & 0 & 0 & 3x_D \\ 0 & 0 & 3x_C & 0 \end{bmatrix}.$$

The payoff shares

$$\mathbf{x} = (x_A, x_B, x_C, x_D) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$$

are clearly an equity equilibrium under $\mathbb{C}^S$ and correspond to the unique Shapley value consistent outcome for $\boldsymbol{\beta}$. $\Diamond$

[18] Recall: A game (N, v) is *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

for all S, $T \subseteq N$.

[19] See Shapley (1971).

The N-Person Generalized Nash Bargaining Solution and the Equity Equilibrium

In this section we provide a short equity equilibrium characterization of the generalized Nash bargaining solution with weights $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$, hereafter GNBS-$\boldsymbol{\alpha}$.

An IRPC $(\mathbf{x}, \boldsymbol{\beta})$ is *consistent* with the GNBS-$\boldsymbol{\alpha}$ if, for each coalition $\beta_j \in \boldsymbol{\beta}$ such that $v(\beta_j) > 0$, the payoff shares maximize the generalized Nash product $\Pi_{i \in \beta_j} x_i^{\alpha_i}$, where $\alpha_i > 0$ for all i.[20]

Let the Nash complaint system, denoted $\mathbb{C}^N(\mathbf{x}, \boldsymbol{\beta})$, be an $n \times n$ matrix whose (k,l)-th element is

$$c_{k,l}^N(\mathbf{x}, \boldsymbol{\beta}) = \begin{cases} v(\beta_j)\left(x_l - \frac{\alpha_l}{\sum_{m \in \beta_j} \alpha_m}\right) & \text{-if } l \text{ \& } k \text{ are in the same coalition } \beta_j \text{, and} \\ 0 & \text{-otherwise.} \end{cases}$$

Clearly, $\mathbb{C}^N$ is a valid *threshold* complaint system. Proposition 4 provides the equity equilibrium characterization.

Proposition 4: *The IRPC $(\mathbf{x}, \boldsymbol{\beta})$ is consistent with the generalized Nash bargaining solution with weights $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$, if and only if $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium under $\mathbb{C}^N$.*

Example 8: Consider the super-additive game (N, v), where $N = \{A, B, C\}$ and the characteristic function is

$$\begin{aligned} v(\{A,B,C\}) &= 3, \\ v(\{A,B\}) &= 1,\ v(\{A,C\}) = v(\{B,C\}) = 2, \\ v(\{A\}) &= v(\{B\}) = v(\{C\}) = 0. \end{aligned}$$

[20] We normalized the payoffs in the game so that $v(\{i\}) = 0$ for all i. We take this to be i's disagreement point.

The shares $x_A = x_B = \frac{1}{6}$ and $x_C = \frac{2}{3}$ form an equity equilibrium for the grand coalition under the complaint system

$$\mathbb{C}^N = \begin{bmatrix} 0 & 3x_B - \frac{1}{2} & 3x_C - 2 \\ 3x_A - \frac{1}{2} & 0 & 3x_C - 2 \\ 3x_A - \frac{1}{2} & 3x_B - \frac{1}{2} & 0 \end{bmatrix}.$$

The outcome is consistent with the generalized Nash bargaining problem with weights $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = (1, 1, 4)$. $\Diamond$

5 Computation of Equity Equilibrium

Proposition 1 provides a non-constructive proof for the existence of equity equilibrium. However, to apply equity equilibrium, we need a reliable method of computation. In this section, we illustrate how simplicial algorithms can be employed to find approximate equity equilibria when players form the grand coalition. This is an expositional choice. When players are in the grand coalition, the task of finding an equity equilibrium is analogous to finding prices that form a Walrasian equilibrium in a general equilibrium problem. In particular, the payoff shares of the players are drawn from the unit simplex. As computing a Walrasian equilibrium was one of the primary motivations for the development of simplicial algorithms, we thus have a large inventory of algorithms from which to choose. Here, we describe how Kuhn's artificial start algorithm can be used to find an approximate equity equilibrium.[21]

First, we provide the key definitions, terminology, and background theorems used by this algorithm.

[21] When players belong to several coalitions, players' shares $\mathbf{x}$ are no longer found on the unit simplex but rather belong to a simplotope (i.e., a Cartesian product of unit simplices). There are simplicial algorithms for approximating fixed points on a simplotope. We refer the reader to Doup (1988) for a book length treatment of the subject, including several efficient algorithms.

Preliminaries

When players form the grand coalition, their payoff shares are $(x_1, ..., x_n) \in X^N$, where X^N is the $(n-1)$ dimensional *unit* simplex. For the algorithm, we need only consider points $\tilde{x} \in X^N$ that can be expressed as $\tilde{x} = \frac{1}{D}(z_1, z_2, ..., z_n)$, where D is the grid denominator (a positive integer), and $z_1, z_2, ..., z_n$ are non-negative integers that sum to D (i.e., $\frac{1}{D}\sum_{i=1}^{n} z_i = 1$). This collection of points, known as the *regular simplicial subdivision* of X^N, chops up X^N into a sequence of smaller $(n-1)$ dimensional *sub-simplices* whose union is X^N. By choosing a sufficiently large grid denominator, we can make this grid arbitrarily fine. Kuhn's algorithm moves along points in a regular subdivision of X^N.

We can assign each point in our regular subdivision with an integer label from the set $\{1, ..., n\}$. The *labeling rule* L we use is defined by the function $L : X^N \to \{1, ..., n\}$ such that

$$L(\mathbf{x}) = \quad k \quad \text{- if } g_k(\mathbf{x}) \leq 0 \text{ and } x_k > 0,$$

with the proviso that if a vertex qualifies for more than one label, then we choose the label l with the most negative $g_l(\mathbf{x}) \leq 0$.

Remark: *From Kuhn and MacKinnon (1975) we know the label rule L is both well-defined on X^N and proper–that is, each point $x \in X^N$ can be assigned a label and can only receive label k when $x_k > 0$.*[22]

Why is this label rule interesting? The answer has to do with Sperner's famous combinatorial lemma, which can be used to prove Brouwer's fixed

[22] A proof of the remark is as follows: Suppose L was not well-defined for some $\mathbf{x}$. Since $\mathbf{x} \in X^N$, there must be at least one index k such that $x_k > 0$, and, for each such k, we must have $g_k(\mathbf{x}) > 0$ (otherwise L is well-defined). However, if this is the case, we have $\sum_k x_k g_k(\mathbf{x}) > 0$, which contradicts the conclusion of Lemma 2. The proof that L is proper follows from the definition of L, since vertex $\mathbf{x}$ can only receive label k if $x_k > 0$. □

point theorem.[23] A sub-simplex of X^N is said to be completely labeled if the n vertices that define the sub-simplex carry all n distinct labels. Sperner's Lemma states that every properly labeled simplicial subdivision of a simplex must contain an odd number of completely labeled sub-simplices.[24]

From the remark, we know that any simplicial subdivision of X^N coupled with a labeling from L is a proper labeling. From Sperner's Lemma, since the labeling is proper, there exists a sub-simplex in the subdivision with a complete set of labels. Thus, in our problem, in the grid there are n vertices $\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^n \in X^N$ that define a completely labeled sub-simplex in the subdivision such that

$$\begin{aligned} g_1(\mathbf{x}^1) &\leq 0 \text{ and } x_1^1 > 0 \\ g_2(\mathbf{x}^2) &\leq 0 \text{ and } x_2^2 > 0 \\ &\vdots \\ g_n(\mathbf{x}^n) &\leq 0 \text{ and } x_n^n > 0. \end{aligned}$$

If the grid of the subdivision is sufficiently fine, then the vertices $\mathbf{x}^1 \approx \mathbf{x}^2 \approx \cdots \approx \mathbf{x}^n$ are near one another and the barycenter of this sub-simplex provides us with our approximate equilibrium point.

Main Idea Behind Simplicial Algorithms

Imagine the simplex X^N is a house and each sub-simplex of the subdivision is a room. We classify a facet of a sub-simplex as a "door" if it bears $n-1$ distinct labels–say labels $\{1, ..., n-1\}$. Thus, every room in the house will have either 0, 1, or 2 doors.

We are going to describe a procedure to walk through the house and find a room with only a single door. The walk starts by locating a door on the

[23] An interesting primer on Sperner's lemma, Brouwer's fixed point theorem, and simplicial algorithms can be found in Su (1999).

[24] The basic idea of the constructive approach is latent in the work of Cohen (1967) and explicit in the work of Kuhn (1968).

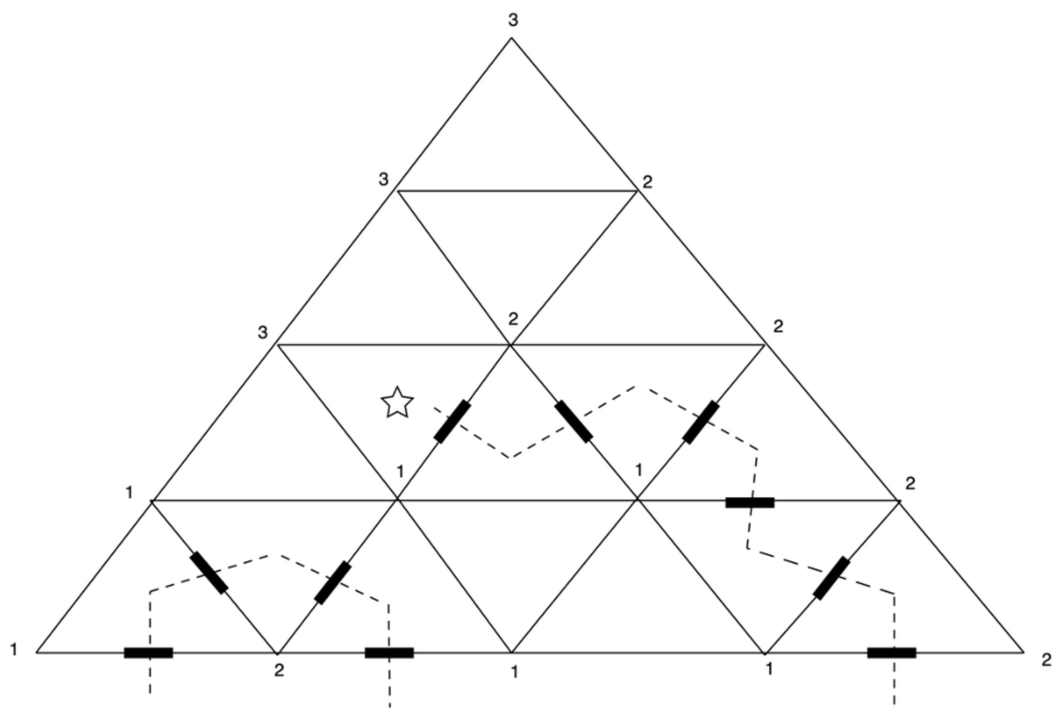

Figure 1: Walking through the House

boundary of the house. This boundary is a facet of X^N associated with labels 1 through $n-1$. By Sperner's lemma we know there are an odd number of doors on this facet. We walk through one of these boundary doors and adopt the rule that we only go through doors that we have not already passed through. By following this rule in each room, we either stop (if there are no more new doors) or we have a unique door left for us to walk through.

There are two ways for such a walk to end: (1) we walk into a room with only 1 door; (2) we walk out of the house. In the first case, we are done. We have found a completely labeled sub-simplex. In the second case, we can simply start again by walking into the house using a different door on the boundary (the existence of such a door is again guaranteed by Sperner's lemma–there are an odd number of such doors and we have only used two!). Following this heuristic, we must eventually go through a boundary door that leads us to a room with only a single door.

Kuhn's algorithm operationalizes this idea into an effective algorithm. A sub-simplex in X^N (i.e., a room in the house) is described by a matrix R. Given a grid size D, the matrix R consists of integers where each column represents a vertex of X^N and sums to D. We use the label rule to assign each vertex in X^N with a label.

The algorithm provides two essential tricks. First, we add an artificial layer (a facet) to the boundary of the house; and then we use a special label rule for those vertices on this artificial facet. The label rule is chosen so that: (1) no completely labeled sub-simplex will appear on the artificial layer of the simplex; and (2) there is precisely 1 door on the boundary of the artificial facet. Thus, there is now a single start to the walking algorithm and it is impossible to exit the house. The algorithm must finish in a room with only a single door–a completely labeled sub-simplex. The second trick is that by using a regular subdivision of X^N, the mathematical operation of walking from one room in the house to another (or by removing one vertex of R and adding a new vertex to arrive in the adjacent sub-simplex R') is captured with a simple algebraic operation involving only 3 columns in R!

In the next section, we provide a detailed step-by-step description of Kuhn's algorithm. We illustrate its use by computing the kernel and the Shapley value of a well-known example, involving the Tennessee Valley Authority.

Kuhn's Artificial Start Algorithm for Finding an Approximate Equity Equilibrium:

We now describe an algorithm for finding a completely labeled sub-simplex.

1. Pick the positive integer D (grid denominator) and identify an integer base vertex $\mathbf{b} = (0, b_2, .., b_n)$, where the sum of the coordinates are equal to D. The base vertex is our initial "entry" vector.

2. Given $\mathbf{b}$, compute the starting sub-simplex whose columns represent the n vertices of the sub-simplex $\mathring{R} = [\mathbf{b}\ \mathbf{x}^2\ \cdots\ \mathbf{x}^n]$, where

$$\mathring{R} = \begin{bmatrix} 0 & -1 & \cdots & -1 & -1 \\ b_2 & b_2+1 & & b_2 & b_2 \\ b_3 & b_3 & & \vdots & \vdots \\ \vdots & \vdots & & b_{n-1}+1 & b_{n-1} \\ b_n & b_n & \cdots & b_n & b_n+1 \end{bmatrix}.$$

The $n-1$ vertices $\mathbf{x}^2\ \cdots\ \mathbf{x}^n$ are not in the unit simplex X^N. We generate labels for each vertex using the rule

$$\tilde{L}(\mathbf{x}) = \begin{cases} L(\mathbf{x}) & \text{- if } \mathbf{x} \in X^N \\ L^a(\mathbf{x}) & \text{- if } \mathbf{x} \notin X^N, \end{cases}$$

where $L(\mathbf{x})$ is defined as earlier and $L^a(\mathbf{x})$ is the labeling rule that assigns $\mathbf{x} \notin X^N$ the smallest integer j such that $x_j > b_j$. For the initial sub-simplex, this rule gives

Vertex:	$\mathbf{b}$	$\mathbf{x}^2$	$\mathbf{x}^3$	$\cdots$	$\mathbf{x}^n$
Label:	$L(\mathbf{b})$	2	3	$\cdots$	n

The starting sub-simplex is thus always "'almost completely labeled" (i.e., has exactly $n-1$ distinct labels).

3. Remove the vector $\mathbf{x}^j$ whose label is the same as the entry vector. The replacement and new entering vector, $\tilde{\mathbf{x}}^j$ is given by

$$\tilde{\mathbf{x}}^j = \mathbf{x}^{j-1} + \mathbf{x}^{j+1} - \mathbf{x}^j \ (mod\ n).$$

4. Compute the label for the entering variable, using $\tilde{L}$. If its label is 1, then stop. A completely labeled sub-simplex has been found. Otherwise, the new sub-simplex is again almost completely labeled and we return to step 3.

Example: Cost Attribution Problem for the Tennessee Valley Authority

In this section, we use Kuhn's algorithm to numerically compute various equity equilibria for the cost attribution problem faced by the Tennessee Valley Authority (TVA) during the 1930s. The TVA project, part of the New Deal, was tasked with improving navigation, controlling flooding, generating electricity, and fostering economic development in the seven states spanned by the Tennessee River Basin. The TVA's goal was to stimulate growth, create jobs, and improve the quality of life for the residents of the area. Economists performing the cost-benefit analysis for the project faced the problem of how much cost should be attributed to each service in the project. They estimated the cost of providing each service by itself, the cost of providing any two services, and the cost of providing all three services. The result was essentially a cooperative game and various methods of assigning costs to services were explored.[25]

[25] Young (1994), pg. 86, provides more details. In particular, he reports that the core of the cooperative game was suggested as a method of cost sharing for the TVA project.

We will use the estimated costs in order to compute cooperative game theory solutions for the TVA problem.

The TVA Cost Attribution Game:

The following table gives the costs c for each subset S of services (i.e., $c(S)$ for services in S) and, in the row below, the associated savings v, where $v(S) = \sum_{i \in S} c(i) - c(S)$.

Services	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Costs $c(S)$	0	163,520	140,826	250,096	301,607	378,370	367,370	412,584
Savings $v(S)$	0	0	0	0	2,739	34,795	23,552	141,858

The cost savings version of the TVA game fits the framework we have used in this paper. We will therefore use it for our computations and then convert the answer into the associated solution for the cost problem. In particular, choosing a vector of savings (x_1, x_2, x_3) is equivalent to choosing costs (c_1, c_2, c_3), where

$$c_i = c(i) - x_i.$$

We call the game $(\{1,2,3\}, v)$ the TVA Game.

The Kernel/Pre-kernel/Nucleolus of the TVA Game:

We begin with the kernel. Since the TVA game is convex, it is also super additive. Thus, when the coalition formed is the grand coalition, the kernel is identical to the pre-kernel and the nucleolus.[26] To compute the complaint

[26] See Corollary 5.7.9 of Peleg and Sudhölter (2007).

system we need to first identify the excess functions for each coalition:

$$
\begin{aligned}
e_1(\mathbf{x}) &= -141858x_1 \\
e_2(\mathbf{x}) &= -141858x_2 \\
e_3(\mathbf{x}) &= -141858x_3 \\
e_{12}(\mathbf{x}) &= 2739 - 141858(x_1 + x_2) \\
e_{13}(\mathbf{x}) &= 34795 - 141858(x_1 + x_3) \\
e_{23}(\mathbf{x}) &= 23552 - 141858(x_2 + x_3)
\end{aligned}
$$

The Maschler-Davis complaint system is then

$$
\begin{aligned}
c^{MD}_{(i,i)}(\mathbf{x}) &= 0, \text{ for } i = 1,2,3 \\
c^{MD}_{(1,2)}(\mathbf{x}) &= \max\{-141858x_1, 34795 - 141858x_1 - 141858x_3\} \\
c^{MD}_{(1,3)}(\mathbf{x}) &= \max\{-141858x_1, 2739 - 141858x_1 - 141858x_2\} \\
c^{MD}_{(2,1)}(\mathbf{x}) &= \max\{-141858x_2, 23552 - 141858x_2 - 141858x_3\} \\
c^{MD}_{(2,3)}(\mathbf{x}) &= \max\{-141858x_2, 2739 - 141858x_1 - 141858x_2\} \\
c^{MD}_{(3,1)}(\mathbf{x}) &= \max\{-141858x_3, 23552 - 141858x_2 - 141858x_3\} \\
c^{MD}_{(3,2)}(\mathbf{x}) &= \max\{-141858x_3, 34795 - 141858x_1 - 141858x_3\}
\end{aligned}
$$

This complaint system defines the excess concession functions:

$$
\begin{aligned}
g_1(\mathbf{x}) &= (c^{MD}_{(1,2)}(\mathbf{x}) - c^{MD}_{(2,1)}(\mathbf{x}))x_2 + (c^{MD}_{(1,3)}(\mathbf{x}) - c^{MD}_{(3,1)}(\mathbf{x}))x_3 \\
g_2(\mathbf{x}) &= (c^{MD}_{(2,1)}(\mathbf{x}) - c^{MD}_{(1,2)}(\mathbf{x}))x_1 + (c^{MD}_{(2,3)}(\mathbf{x}) - c^{MD}_{(3,2)}(\mathbf{x}))x_3 \\
g_3(\mathbf{x}) &= (c^{MD}_{(3,1)}(\mathbf{x}) - c^{MD}_{(1,3)}(\mathbf{x}))x_1 + (c^{MD}_{(3,2)}(\mathbf{x}) - c^{MD}_{(2,3)}(\mathbf{x}))x_2
\end{aligned}
$$

The table below describes the results of Kuhn's algorithm for grid denominator $D = 10, 100, 1000, 10000$, and 100000.[27]

		Approx. Equity Eq. ($\mathbb{C}^{MD}$)		
Grid D	*Flips*	x_1	x_2	x_3
10	12	0.3333	0.2333	0.4333
100	132	0.3333	0.3233	0.3433
1000	1332	0.3333	0.3323	0.3343
10000	13334	0.3333	0.3332	0.3334
100000	133350	0.3333	0.3333	0.3333

Direct computation verifies that $(x_1, x_2, x_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is indeed in the kernel/pre-kernel/nucleolus of the TVA Game. We can convert this solution to the cost attribution game as follows:

$$\begin{aligned} c_1 &= 163520 - \frac{141858}{3} = 116\,234 \\ c_2 &= 140826 - \frac{141858}{3} = 93\,540 \\ c_3 &= 250096 - \frac{141858}{3} = 202\,810. \end{aligned}$$

By construction these costs add up to the total cost of the joint project $116\,234 + 93\,540 + 202\,810 = 412\,584$ and yield the following cost shares associated with the kernel

	1	2	3
Cost Share (kernel)	0.281 7	0.226 7	0.491 6

These cost shares numbers are identical to those found in Straffin and Heaney's computation of the nucleolus.

The Shapley Value of the TVA Game:

Next, we compute the Shapley value of the TVA game. For the three player game, it is easy to directly compute the Shapley value using the marginal benefit method of adding a player to the grand coalition. Specifically,

[27] In each case, the program spent less than a second to run on the author's laptop.

for each of the $|N|!$ possible permutations of players, we compute the marginal value of adding each player to the coalition (in the given order). The Shapley value is the average of these marginal benefits taken over all possible orderings of the players. The table below computes the Shapley value in this manner. The numbers given in the last row are the averages of their respective columns.

	1	2	3
123	0	2739	139 119
132	0	107 063	34795
213	2739	0	139 119
231	118 306	0	23552
312	34795	107 063	0
321	118 306	23552	0
Shapley:	45691	40069.5	56097.5

The Shapley value shares are therefore

$$
\begin{aligned}
x_1 &= \frac{45691}{141858} = 0.32209 \\
x_2 &= \frac{40069.5}{141858} = 0.28246 \\
x_3 &= \frac{56097.5}{141858} = 0.39545
\end{aligned}
$$

for players 1, 2, and 3, respectively.

Next, we use the Kuhn algorithm to compute the Shapley value as a check on our process. The Shapley dividend for each subcoalition is the equity equilibrium complaint for that subgame. As a consequence, the dividends can be computed recursively, using Kuhn's algorithm in each step. The dividends, for coalitions smaller than the grand coalition, in the TVA Game are

$$
\begin{aligned}
w_1 &= w_2 = w_3 = 0 \\
w_{12} &= 1369.\,5 \\
w_{13} &= 17397.5 \\
w_{23} &= 11776
\end{aligned}
$$

The Shapley complaint system for the grand coalition is therefore

$$
\begin{aligned}
c^S_{(i,i)}(\mathbf{x}) &= 0, \text{ for } i = 1,2,3 \\
c^S_{(1,2)}(\mathbf{x}) &= 141858x_2 - 13145.5 \\
c^S_{(1,3)}(\mathbf{x}) &= 141,858x_3 - 29173.5 \\
c^S_{(2,1)}(\mathbf{x}) &= 141,858x_1 - 18767 \\
c^S_{(2,3)}(\mathbf{x}) &= 141,858x_3 - 29173.5 \\
c^S_{(3,1)}(\mathbf{x}) &= 141,858x_1 - 18767 \\
c^S_{(3,2)}(\mathbf{x}) &= 141,858x_2 - 13145.5
\end{aligned}
$$

The table below gives the approximate equity equilibrium for this complaint system for grid sizes $D = 10, 100, 1000, 10000,$ and 100000. Again, each iteration of the algorithm took less than a second on the author's laptop. The accuracy of the approximation is already correct to two decimal points at $D = 100$.

		Approx. Equity Eq. ($\mathbb{C}^S$)		
Grid D	*Flips*	x_1	x_2	x_3
10	16	0.3333	0.2333	0.4333
100	154	0.3233	0.2833	0.3933
1000	1532	0.3223	0.2823	0.3953
10000	15312	0.3220	0.2824	0.3955
100000	153120	0.3221	0.2825	0.3954

The implied costs of the Shapley value are

$$\begin{aligned} c_1 &= 163520 - (0.3221)141858 = 117827.5382 \\ c_2 &= 140826 - (0.2825)\,141858 = 100751.115 \\ c_3 &= 250096 - (0.3954)\,141858 = 194005.3468 \end{aligned}$$

By construction, these costs add up to the total cost of the joint project and yield the following cost shares associated with the Shapley value:

	1	2	3
Cost Share (Shapley)	0.2855	0.2442	0.4702

These cost shares match those we computed earlier and those computed by Straffin and Heany (1980) for the TVA Cost Sharing game.

6 Discussion

Equity equilibria are a useful way to think about solutions in cooperative games. They exist. They can be used to characterize disparate classic solutions. And they are straightforward to compute using simplicial algorithms. However, our results only scratch the surface as it is clear that many additional relationships between equity equilibria and classic cooperative solution concepts are possible. As an example, we obtained the following equity equilibrium, which is always contained in the epsilon core.

Consider a game (N, v) where the coalition of interest is the grand coalition. Then define a complaint system $\mathbb{C}^{\epsilon}$, where k's complaint against each other player l is given by

$$\begin{aligned} c^{\epsilon}_{(k,l)}(\mathbf{x}, N) &= \min \epsilon \text{ such that:} \\ \epsilon &\geq v(S) - \sum_{i \in S} x_i v(N) \text{ for all } S \subset N \text{ such that } k \in S. \\ \epsilon &\geq 0 \end{aligned}$$

In other words, $c^{\epsilon}_{(k,l)}(\mathbf{x},N)$ is the amount that k needs so that they have no profitable coalitional deviations.

The out-complaints relation $\rhd$ induced by $\mathbb{C}^{\epsilon}$ is transitive. Thus, from the characterization lemma (i.e., Lemma 3), if $(\mathbf{x}, N)$ is an equity equilibrium, we have: that if $c_{k,l}(\mathbf{x}, N) > c_{l,k}(\mathbf{x}, N)$, then $x_l = 0$; and, for any pair k, $l \in N$ such that $x_k > 0$ and $x_l > 0$, we have

$$c_{k,l}(\mathbf{x}, N) = c_{l,k}(\mathbf{x}, N).$$

It follows that, for any k such that $x_k > 0$, the complaint $c^{\epsilon}_{(k,j)}(\mathbf{x}, \boldsymbol{\beta}) = \epsilon^*$ defines an ϵ^*-core and the equity equilibrium $(\mathbf{x},N)$ belongs to this set. Thus, all equity equilibria under $\mathbb{C}^{\epsilon}$ belong to an epsilon core defined by the magnitudes of the complaints of the players receiving positive shares.

This result, while interesting, has two shortcomings: First, it is not a full characterization of the epsilon core. It is not clear whether all epsilon core outcomes can be supported as an equity equilibrium under $\mathbb{C}^{\epsilon}$. Thus, we only know that equity equilibria are found in *an* epsilon core. Second, the epsilon core outcome identified by an equity equilibrium has a unique structure (all players receiving positive shares have the same complaint ϵ). However, at this time, we do not know whether the equity equilibria under $\mathbb{C}^{\epsilon}$ are otherwise special. For example, do equity equilibria under this complaint system select a core outcome (if available), or a least core outcome. Resolving this open question would be interesting for future research.

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A. Appendix

In this appendix we provide the formal proofs for the results presented in the body of the paper.

Proof of Lemma 2: We need to show, for each coalition $\beta_j \in \boldsymbol{\beta}$,

$$\sum_{l \in \beta_j} x_l g_l(\mathbf{x}) = 0.$$

Choosing $l \in \beta_j$ and writing out $x_l g_l(\mathbf{x})$, we have

$$\begin{aligned} x_l g_l(\mathbf{x}) &= x_l x_1 \left(c_{l,1}(\mathbf{x}) - c_{1,l}(\mathbf{x})\right) + x_l x_2 \left(c_{l,2}(\mathbf{x}) - c_{2,l}(\mathbf{x})\right) + ... \\ &\quad + x_l x_n \left(c_{l,n}(\mathbf{x}) - c_{n,l}(\mathbf{x})\right). \end{aligned}$$

In the expression $x_l g_l(\mathbf{x})$, for pairs $l, k \in \beta_j$, we have the term $x_l x_k \left(c_{l,k}(\mathbf{x}) - c_{k,l}(\mathbf{x})\right)$. Also, in the expression $x_k g_k(\mathbf{x})$, we have the term $x_k x_l \left(c_{k,l}(\mathbf{x}) - c_{l,k}(\mathbf{x})\right)$; and these terms offset each other when added together, since

$$x_l x_k \left(c_{l,k}(\mathbf{x}) - c_{k,l}(\mathbf{x})\right) + x_k x_l \left(c_{k,l}(\mathbf{x}) - c_{l,k}(\mathbf{x})\right) = 0.$$

Hence, in the sum $\sum_{l \in \beta_j} x_l g_l(\mathbf{x})$ all the terms corresponding to pairs of players in β_j offset each other. Next, since the complaint system is valid, for each term corresponding to $l \in \beta_j$ and $k \notin \beta_j$,

$$x_l x_k \left(c_{l,k}(\mathbf{x}) - c_{k,l}(\mathbf{x})\right) = 0.$$

Finally, for each $l \in \beta_j$, the term $x_l x_l \left(c_{l,l}(\mathbf{x}) - c_{l,l}(\mathbf{x})\right) = 0$. It follows $\sum_{l \in \beta_j} x_l g_l(\mathbf{x}) = 0$. ■

Proof of Proposition 1: Suppose $\boldsymbol{\beta} = \{\beta_1, ..., \beta_m\}$, where each coalition β_j's payoff shares $(x_{ji})_{i \in \beta_j}$ belong to the appropriate unit simplex X^{β_j}. Define the transition function $F : \times_{j=1}^m X^{\beta_j} \to \times_{j=1}^m X^{\beta_j}$, where $F(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_m(\mathbf{x}))$ and $f_j(\mathbf{x}) = \left(f_{j1}(\mathbf{x}), ..., f_{j|\beta_j|}(\mathbf{x})\right)$, where for each $l \in \beta_j$,

$$f_{jl}(\mathbf{x}) = \frac{x_{jl} + \max\{0, g_l(\mathbf{x})\}}{1 + \sum_{k \in \beta_j} \max\{0, g_k(\mathbf{x})\}}$$

for $j = 1, ..., m$. From Lemma 1, the function F is continuous. It is routine to check that F maps a non-empty, compact, and convex set $\times_{j=1}^{m} X^{\beta_j}$ back into itself and is continuous. Hence, by Brouwer's fixed point theorem there exists an $\mathbf{x}^* \in \times_{j=1}^{m} X^{\beta_j}$ such that $F(\mathbf{x}^*) = \mathbf{x}^*$.

At this fixed point, for each coalition j and player l we have

$$x_{jl}^* = \frac{x_{jl}^* + \max\{0, g_l(\mathbf{x}^*)\}}{1 + \sum_{k \in \beta_j} \max\{0, g_k(\mathbf{x}^*)\}}.$$

Simplifying the above expression gives

$$\max\{0, g_l(\mathbf{x}^*)\} = x_{jl}^* \left(\sum_{k \in \beta_j} \max\{0, g_k(\mathbf{x}^*)\} \right).$$

Next, we multiply both sides by $g_l(\mathbf{x}^*)$ and sum over $l \in \beta_j$

$$\sum_{l \in \beta_j} g_l(\mathbf{x}^*) \max\{0, g_l(\mathbf{x}^*)\} = \left(\sum_{k \in \beta_j} \max\{0, g_k(\mathbf{x}^*)\} \right) \left(\sum_{l \in \beta_j} x_{jl}^* g_l(\mathbf{x}^*) \right).$$

From Lemma 2, since the complaint system is valid, the above simplifies to

$$\sum_{l \in \beta_j} g_l(\mathbf{x}^*) \max\{0, g_l(\mathbf{x}^*)\} = 0.$$

Since each addend is non-negative, for each $l \in \beta_j$ we must have that $g_l(\mathbf{x}^*) \leq 0$.

From Lemma 2, since each $x_{jl}^* \geq 0$ and each $g_l(\mathbf{x}^*) \leq 0$, we have that $x_{jl}^* g_l(\mathbf{x}^*) = 0$. Since this is true for all players and all coalitions, $(\mathbf{x}^*, \boldsymbol{\beta})$ is an equity equilibrium. ■

Proof of Lemma 3: Proof of Part (i): Suppose the out-complaints relation $\triangleright$ induced by $\mathbb{C}$ is transitive and that $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium associated with $\mathbb{C}$. Let k and l belong to the same coalition β_j with

$$c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) > c_{l,k}(\mathbf{x}, \boldsymbol{\beta}).$$

We need to show that $x_l = 0$.

Suppose not. If $|\beta_j| = 2$ and $x_l > 0$, then k's total concession function is

$$g_k(\mathbf{x}, \boldsymbol{\beta}) = (c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) - c_{l,k}(\mathbf{x}, \boldsymbol{\beta}))\, x_l.$$

However, $g_k > 0$ contradicts that we are in an equity equilibrium.

Suppose, $|\beta_j| \geq 3$ and $x_l > 0$. Since $c_{k,l} > c_{l,k}$ and $x_l > 0$, we have that $k \rhd l$. However, we are in an equity equilibrium. The concession to k in the amount $(c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) - c_{l,k}(\mathbf{x}, \boldsymbol{\beta}))\, x_l > 0$ must be offset by grievances. Thus, there must be at least one other player $j_1 \neq l, k$ such that

$$c_{k,j_1}(\mathbf{x}, \boldsymbol{\beta}) < c_{j_1,k}(\mathbf{x}, \boldsymbol{\beta})$$

and $x_{j_1} > 0$. Since $c_{j_1,k} > c_{k,j_1}$ and $x_{j_1} > 0$, we have that $j_1 \rhd k$. From transitivity of $\rhd$ relation, $j_1 \rhd l$.

Next $j_1 \rhd l$ and $x_l > 0$ imply that l makes a concession to j_1 in the amount $(c_{j_1,l}(\mathbf{x}, \boldsymbol{\beta}) - c_{l,j_1}(\mathbf{x}, \boldsymbol{\beta}))\, x_l > 0$. Since we are in an equity equilibrium this concession must be offset by grievances against j_1. Thus, there needs to be at least one other player $j_2 \neq l, k, j_1$ such that

$$c_{j_1,j_2}(\mathbf{x}, \boldsymbol{\beta}) < c_{j_2,j_1}(\mathbf{x}, \boldsymbol{\beta})$$

and $x_{j_2} > 0$. So, $j_2 \rhd j_1$. By the transitivity of $\rhd$, we have $j_2 \rhd k$ and $j_2 \rhd l$.

We continue this process. At the end of step $m-1$, we conclude that $j_{m-1} \rhd l$ and $x_l > 0$. In step m, since we are in equilibrium, we must have another player j_m distinct from l, k and $, j_1, ..., j_{m-1}$ with $j_m \rhd j_{m-1}$ and $x_{j_m} > 0$. By transitivity, we conclude that $j_m \rhd l$ with $x_l > 0$.

This process continues until eventually we run out of people in β_j. This contradicts the claim that we are in an equity equilibrium. Hence, $c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) > c_{l,k}(\mathbf{x}, \boldsymbol{\beta})$ implies $x_l = 0$. $\square$

The proofs of parts (ii) and (iii) follow immediately from part (i). $\blacksquare$

Proof of Lemma 4:

(Proof of 4.i): First, $\rhd$ is not reflexive. By definition of $\mathbb{C}$, $c_{k,k}(\mathbf{x}, \boldsymbol{\beta}) = 0$ for all k. So, we can't have $k \rhd k$.

Second, if $k \rhd l$, then $c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) > c_{l,k}(\mathbf{x}, \boldsymbol{\beta})$ and $x_l > 0$. Clearly, we cannot also have $l \rhd k$, since then $c_{k,l}(\mathbf{x}, \boldsymbol{\beta}) < c_{l,k}(\mathbf{x}, \boldsymbol{\beta})$, which is not possible.

Finally, we establish transitivity. Choose $a, b, c \in \boldsymbol{\beta}$. Suppose $a \rhd b$ and $b \rhd c$. We need to show that $a \rhd c$.

Since $a \rhd b$ and $b \rhd c$, we have

$$\begin{aligned} c_{a,b} - c_{b,a} &> 0 \\ c_{b,c} - c_{c,b} &> 0 \end{aligned}$$

with $x_b > 0$ and $x_c > 0$. Since $\mathbb{C}$ is a threshold complaint system, we have $c_{b,c} = c_{a,c}$ and $c_{b,a} = c_{c,a}$. Further, since $c_{c,b} = c_{a,b} > c_{b,a} = c_{c,a}$, we have

$$c_{b,c} - c_{c,b} = c_{a,c} - c_{c,b} > c_{a,c} - c_{c,a} > 0.$$

Thus, we have $c_{a,c} - c_{c,a} > 0$ and $x_c > 0$ so $a \rhd c$. $\square$

(Proof of 4.ii) Assume $v(\beta_j) > 0$ for each coalition j. The result is trivial for $|\beta_j| = 1$. Therefore assume $|\beta_j| \geq 2$. Suppose IRPC $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium under $\mathbb{C}(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b})$, where there exists an IRPC $(\bar{\mathbf{x}}, \boldsymbol{\beta})$ *such that, for each coalition* β_j, we have $c_{k,l}(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b}) = c_{l,k}(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b})$ *for all* $l,k \in \beta_j$. Suppose player l's equilibrium share x_l is different from $\bar{x}_l$. Without loss, assume $x_l < \bar{x}_l$. Since the shares of the coalition need to add to one, this implies that there is a player k with $x_k > \bar{x}_k$. Since $v(\beta_j) > 0$, we have

$$\begin{aligned} c_{l,k}(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b}) &= v(\beta_j)x_k - b_k \\ &> v(\beta_j)\bar{x}_k - b_k \\ &= v(\beta_j)\bar{x}_l - b_l \\ &> v(\beta_j)x_l - b_l = c_{k,l}(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b}). \end{aligned}$$

But then

$$x_k(c_{l,k}(\mathbf{x}, \boldsymbol{\beta}) - c_{k,l}(\mathbf{x}, \boldsymbol{\beta})) > 0.$$

This is a contradiction since the inequality violates Lemma 3.*ii*.

Now suppose $v(\beta_j) = 0$. Again suppose player l's equilibrium share x_l is different from $\bar{x}_l$. Without loss, assume $x_l < \bar{x}_l$. Since the shares of the coalition need to add to one, this implies that there is a player k with $x_k > \bar{x}_k$.

$$\begin{aligned} c_{l,k}(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b}) &= v(\beta_j)x_k - b_k \\ &= -b_k \\ &= v(\beta_j)\bar{x}_k - b_k \\ &= v(\beta_j)\bar{x}_l - b_l \\ &= -b_l \\ &= v(\beta_j)x_l - b_l = c_{k,l}(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b}). \end{aligned}$$

It follows that $c_{l,k}(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b}) = c_{k,l}(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b}) = c_{k,l}(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b})$ at all equity equilibria.

■

Proof of Proposition 2: Suppose $(\mathbf{x}, \boldsymbol{\beta}) \in K$. We first need to show that, for each k, we have either $g_k(\mathbf{x}) = 0$, or $g_k(\mathbf{x}) < 0$ and $x_k = 0$.

Since $(\mathbf{x}, \boldsymbol{\beta}) \in K$ and suppose $k \in \beta_j$, then since $(\mathbf{x}, \boldsymbol{\beta}) \in K$, we have

$$(c^{MD}_{(k,l)}(\mathbf{x}, \boldsymbol{\beta}) - c^{MD}_{(l,k)}(\mathbf{x}, \boldsymbol{\beta}))x_l \leq 0$$

for all $l \in \beta_j$. Also, since $\mathbb{C}^{MD}$ is a valid complaint system, $c^{MD}_{(k,l)}(\mathbf{x}, \boldsymbol{\beta}) = c^{MD}_{(l,k)}(\mathbf{x}, \boldsymbol{\beta}) = 0$ for all $l \notin \beta_j$. Adding up these inequalities gives us

$$\begin{aligned} g_k(\mathbf{x}) &= x_1\left(c^{MD}_{(k,1)}(\mathbf{x}, \boldsymbol{\beta}) - c^{MD}_{(1,k)}(\mathbf{x}, \boldsymbol{\beta})\right) + \\ &\quad x_2\left(c^{MD}_{(k,2)}(\mathbf{x}, \boldsymbol{\beta}) - c^{MD}_{(2,k)}(\mathbf{x}, \boldsymbol{\beta})\right) + \\ &\quad ... + \\ &\quad x_n\left(c^{MD}_{(k,n)}(\mathbf{x}, \boldsymbol{\beta}) - c^{MD}_{(n,k)}(\mathbf{x}, \boldsymbol{\beta})\right). \\ &\leq 0. \end{aligned}$$

From Lemma 2,

$$\sum_{k \in \beta_j} x_k g_k(\mathbf{x}) = 0.$$

Since each $g_k(\mathbf{x}^*) \leq 0$ and each $x_k^* \geq 0$, we must have (1) if $x_k^* > 0$, then $g_l(\mathbf{x}^*) = 0$; and (2) if $g_k(\mathbf{x}^*) < 0$, then $x_k = 0$. Thus, $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium.

Now suppose $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium under $\mathbb{C}^{MD}$. Since $\mathbb{C}^{MD}$ induces a transitive out-complaints relation $\rhd$ (Maschler-Davis), Lemma 3.1 (the characterization lemma) implies that if $c_{k,l}^{MD}(\mathbf{x}, \boldsymbol{\beta}) > c_{l,k}^{MD}(\mathbf{x}, \boldsymbol{\beta})$, then $x_l = 0$. Thus, $(\mathbf{x}, \boldsymbol{\beta}) \in K$. ■

Proof of Proposition 3: Suppose $(\mathbf{x}, \boldsymbol{\beta})$ is Shapley consistent for $\boldsymbol{\beta}$. First, since the game is subgame super-additive under $\boldsymbol{\beta}$, the outcome $(\mathbf{x}, \boldsymbol{\beta})$ is individually rational. Also, since Shapley consistent shares sum to one within their coalition, we have that $(\mathbf{x}, \boldsymbol{\beta})$ is an IRPC. Now, consider player $k \in \beta_j$. Since $\phi_l(v|_{\beta_j}) = v(\beta_j)x_l$ and $\phi_l(v|_{\beta_j}) = \sum_{l \in R, R \subseteq \beta_j} w^R$, we have $c_{k,l}^S(\mathbf{x}, \boldsymbol{\beta}) = w_{\beta_j}$ for $l \in \beta_j$. Also, by the same argument, for $l \in \beta_j$, we have $c_{l,k}^S(\mathbf{x}, \boldsymbol{\beta}) = w_{\beta_j}$. It follows that $c_{l,k}^S(\mathbf{x}, \boldsymbol{\beta}) = c_{k,l}^S(\mathbf{x}, \boldsymbol{\beta})$ for all $l, k \in \beta_j$. Since the complaint system is valid, all other complaints are zero. Hence, $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium under $\mathbb{C}^S$.

Next, suppose $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium under $\mathbb{C}^S$. Since $\mathbb{C}^S$ is a threshold complaint system, the results from Lemma 4 apply. The game (N, v) is subgame super-additive under $\boldsymbol{\beta}$. There are Shapley value consistent shares $\bar{\mathbf{x}} \in X$ such that $(\bar{\mathbf{x}}, \boldsymbol{\beta})$ is an IRPC and, for each coalition β_j, we have $c_{k,l}^S(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b}) = c_{l,k}^S(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b})$ for all $l,k \in \beta_j$. Thus, from Lemma 4.ii, *every* equity equilibrium $(\mathbf{x}, \boldsymbol{\beta})$ under $\mathbb{C}^S$ for the partition $\boldsymbol{\beta}$ has $c_{k,l}^S(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b}) = c_{l,k}^S(\mathbf{x}, \boldsymbol{\beta}; \mathbf{b}) = c_{k,l}^S(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b}) = w_{\beta_j}$ for all $l,k \in \beta_j$. We therefore have

$$\begin{aligned} v(\beta_j)x_l &= c_{k,l}^S(\mathbf{x}, \boldsymbol{\beta}) + \sum_{l \in R, R \subset \beta_j} w^R \\ &= w_{\beta_j} + \sum_{l \in R, R \subset \beta_j} w^R \\ &= \phi_l(v|_{\beta_j}). \end{aligned}$$

for each $l \in \beta_j$. The equilibrium payoff is their Shapley value consistent payoff. ■

Proof of Proposition 4: Suppose $(\mathbf{x}, \boldsymbol{\beta})$ is consistent with the generalized Nash bargaining problem under $\boldsymbol{\alpha}$ for $\boldsymbol{\beta}$. Then

$$x_l = \frac{\alpha_l}{\sum_{m \in \beta_j} \alpha_m}$$

for all $l \in \beta_j$. Hence, $c^N_{l,k}(\mathbf{x}, \boldsymbol{\beta}) = c^N_{k,l}(\mathbf{x}, \boldsymbol{\beta}) = 0$ for all $l, k \in \beta_j$. Since the complaint system is valid, all other complaints are zero. Hence, $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium under $\mathbb{C}^N$.

Next, suppose $(\mathbf{x}, \boldsymbol{\beta})$ is an equity equilibrium under $\mathbb{C}^N$. Since $\mathbb{C}^N$ is a valid threshold complaint system, the results from Lemma 4 apply. The generalized Nash bargaining shares $\bar{\mathbf{x}} \in X$ such that $(\bar{\mathbf{x}}, \boldsymbol{\beta})$ is an IRPC and, for each coalition β_j, we have $c^N_{k,l}(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b}) = c^N_{l,k}(\bar{\mathbf{x}}, \boldsymbol{\beta}; \mathbf{b})$ for all $l,k \in \beta_j$. Thus, from Lemma 4.ii, *every* equity equilibrium $(\mathbf{x}, \boldsymbol{\beta})$ under $\mathbb{C}^N$ for the partition $\boldsymbol{\beta}$ has $c^N_{k,l}(\mathbf{x}, \boldsymbol{\beta}) = c^N_{l,k}(\mathbf{x}, \boldsymbol{\beta}) = c^N_{k,l}(\bar{\mathbf{x}}, \boldsymbol{\beta}) = 0$ for all $l,k \in \beta_j$. Moreover, for each coalition where $v(\beta_j) > 0$, the solution is unique and precisely the solution to the generalized Nash bargaining problem. Hence, the equity equilibrium under $\mathbb{C}^N$ is consistent with the generalized Nash bargaining solution under $\boldsymbol{\alpha}$. ■