Assessing the Economic Tradeoffs Between Prevention and Suppression of Forest Fires

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Abstract

The number of large-scale, high-severity forest fires occurring in the United States is increasing, as is the cost to suppress these fires. One of the key challenges in studying the costs and benefits of forest fire prevention management is the incorporation of risk and uncertainty surrounding management decisions. We use a technique developed by William Reed to incorporate the stochasticity of the time of a forest fire into our optimal control problem. Using this optimal control problem we explore the potential trade-offs between prevention management spending and suppression spending, along with the overall economic viability of prevention management spending. Our goal is to determine the optimal fire prevention management spending rate and the optimal fire suppression spending which maximizes the expected value of a forest. We develop a parameter set reflecting the 2011 Las Conchas Fire in New Mexico and numerically solve our optimal control problem. Furthermore, we adapt this problem to simulate a sequence of fires and corresponding controls. We perform a simulation study to determine how, on average, prevention management spending affects the value of a forest given an unknown number of fires over a fixed management horizon. Overall, our results support the conclusion that the prevention management efforts offset rising suppression costs and increase the value of a forest overall.

1 Introduction

The number of large-scale, high-severity forest fires occurring in the United States is increasing. Despite a decreasing trend for the total number of fires occurring each year, the total number of acres being burned each year is increasing [22]. This suggests that fires are larger and more severe, on average. In 2015, there were 44 large fires that burned over 40,000 acres each [22]. Calkin et al. conclude that only 1% of wildfires account for 97.5% of the total number of acres burned [5].

In addition to increasingly large fires, the cost to suppress, contain, and extinguish these fires is increasing [22, 8]. One explanation for the increase in fire suppression costs includes decades of fire suppression and exclusion policies which have resulted in uncharacteristically continuous and dense forests with more ladder fuels [5]. In the past century there has been an active fire exclusion effort in the United States; this means that wildland fires have not been allowed to burn despite the history and relationship of fire to a given ecosystem or region. As a result, some ecosystems have been significantly altered, leading to more continuous, dense forests which support devastating severe fire events [9, 2]. In particular, fire-adapted ecosystems, where low-intensity surface fires were a common occurrence and were regenerative, now experience high-severity, stand-replacing fires where most of the trees are killed [9]. Other explanations for the increase in fire suppression costs include an expanding wildland-urban interface (WUI), prolonged drought (climate change), and the lack of financial accountability for fire managers [5].

These increasing trends in wildfire size and federal suppression costs have prompted investigations into alternative methods to help prevent and manage these large wildfires. One such alternative is fuels management, defined in the USDA Forest Service Manual as the "practice of controlling flammability and reducing resistance to control of wildland fuels through mechanical, chemical, biological or manual means, or by fire, in support of land management objectives," [18]. Roughly 67 millions acres of forest have left their natural fire regime and are in need of some form of fuels management [33]. The Wildland Fire Strategic Plan: 2015-2019 put forth by the National Park Service emphasizes the importance of determining areas where fuels management treatments are needed and the importance of developing appropriate programs to address these treatments [35].

It is not feasible to experimentally test the impact of fuels treatments on suppression costs on a large scale[2, 38]. However, empirical evidence for the efficacy of fuels treatments to reduce fire hazard and the size of a fire has been observed following several large fire events [2]. Even though evidence in favor of fuels management is growing, fire suppression spending still outweighs expenditures on hazardous fuels reduction [11]. In fact, in extreme fire years emergency funding for fire suppression has been appropriated from funds designated for fuels management programs [37]. Constraints surrounding smoke, endangered species, regulatory review, and lack of societal acceptance inhibit timely implementation of fuel management strategies [37, 34]. Furthermore, there has been limited economic analysis concerning the viability of such fire prevention management strategies [10, 14, 18]. In particular, Mercer et al. [18] stress that, "Two of the most important unanswered economic questions are whether the resources expended to reduce wildfire risk result in net economic gains and how to quantify the trade-offs between increasing expenditures on suppression and fuels management."

Increasingly, researchers are turning to economics to inform wildland fire prevention management plans [20]. Mercer *et al.* [19] use dynamic stochastic programming and a Monte Carlo simulation model to test the impact of alternative prescribed burning applications on the overall welfare of a forest. They conclude that net economic gains may result from extensive prescribed burning in a specified area. However, this model is only applied to one specific area, does not account for ecosystems differences, is scale dependent, and thus, is not easily broadly applied [19]. Additionally, this work does not address the trade-offs between increasing suppression costs and prevention management spending. In a different study, a standard-response model is modified and linear-integer optimization is used to examine the trade-offs between fuels management alternatives and initial wildfire suppression attack resource deployment [18]. Minas *et al.* [21] present an integer programming model which fully integrates fuel treatment and fire suppression planning.

However, none of these studies consider how trade-offs between fire prevention and suppression are shaped by the inherent risk and uncertainty associated with fire events. The benefits of fire prevention are only realized when a fire occurs. Because the timing of fires are unknown, the benefits fire prevention are uncertain. Suppression represents a relatively more certain investment since it works to limit the damages from an existing fire. In a literature review of economic studies exploring the cost and benefits of wildland fires and their management, Milne *et al.* found that one of the key challenges in these studies is the incorporation of risk and uncertainty surrounding management decisions [20]. This work aims to address this challenge by modeling the economic trade-offs between fire prevention management spending and fire suppression spending when the time of fire is unknown.

In the late 1980s and early 1990s Reed wrote a series of papers exploring the management of a resource vulnerable to random catastrophic collapse [26, 27, 28, 29, 6]. Reed developed a method [30] to convert an initially stochastic problem, due to the random time of collapse, into a deterministic optimal control problem where Pontryagin's Maximum Principle may be utilized [31]. This technique has been applied to forestry [26, 27], invasive species [7], and emerging infectious diseases [4, 12]. A summary of Reed's method, along with its different applications, are found in [31].

We use Reed's method to consider optimal prevention spending when the time of fire is unknown. The strength of Reed's method is its ability to incorporate the risk of a significant catastrophic event into resource management models, especially when there is a distinction between control management strategies employed before, during, or after the event. Investment in preventative management before a catastrophic fire is risky because its benefits are realized at some unknown time in the future. Because managers are often risk-averse, they prefer implementing control measures only after the fire has broken out because the cost and benefits of these measures occur at approximately the same time and are relatively more certain [7]. Thus, the Reed method allows us to investigate and quantify how the uncertainty in the timing of large fire events influences preventative management before a fire and the level of suppression management during a fire.

Previous applications of Reed's method consider an infinite time horizon and one catastrophic event. In contrast, our application of Reed's method considers a finite time horizon with an unknown number of catastrophic events by successively applying the optimal control problem to study sequences of fires. We apply our optimal control problem multiple times in succession and sample for the times of fires using a cumulative distribution function built with the solution to our optimal control problem. We perform a simulation study in order to determine the average value of the forest over our management horizon given that an unknown number of fires may occur, and look at the tradeoffs between total prevention management spending and suppression spending. Additionally, our method differs from others in that we explicitly determine a function describing the optimal value of a forest following a fire using scalar optimization to determine optimal suppression spending at the time of fire. This allows us to quantitatively examine the effects of prevention management spending and suppression spending on the overall economic value of a forest. By choosing functional forms and parameter ranges explicitly, we are also able to perform a parameter sensitivity analysis on our optimal control problems to determine which parameters have the most impact on the value of the forest. To our knowledge this type of global sensitivity analysis has not been performed for other problems applying Reed's method.

This work contributes to the fire economics literature because it is the first to use Reed's method to examine how fire risk influences tradeoffs between prevention and suppression. Furthermore, this work is the first to use Reed's method to look at multiple random events and the first to perform a global sensitivity analysis using Latin Hypercube Sampling and partial rank correlation coefficients to rank parameters based on their impact on the value of the objective functional.

In Section 2 the formulation of our optimal control problem is given and we derive the corresponding necessary conditions using Pontryagin's Maximum Principle (PMP). In Section 3 we build a parameter set based on a 2011 fire in New Mexico (the Las Conchas Fire), numerically approximate the solution, and interpret the results. We also perform a global parameter sensitivity analysis, using Latin Hypercube Sampling and partial rank correlation coefficients, to determine the parameters in our problem which have a significant impact on the value of our objective functional and the mean optimal prevention management spending rate. In Section 4, we consider the impact of prevention management spending on the value of a forest for an unknown sequence of fires over a fixed management horizon. We finish with some conclusions.

2 Model Formulation

We want to incorporate the uncertainty surrounding the time of fire into our study as this is one of the key challenges in addressing and developing fire management strategies [20]. Our goal is to determine the optimal time path of prevention expenditures which will maximize the expected net present value of the forest over a finite time horizon. To achieve this goal, we solve the problem using backward induction. First, we solve for the optimal *ex post* fire suppression spending at the time of the fire. Given the optimized value function after the fire occurs, we then solve for the optimal *ex ante* fire prevention spending schedule given the optimized *ex post* value function.

We assume the effects of prevention management spending are instantaneous and that prevention management spending at the time of fire will decrease the number of acres burned in the fire and that it will decrease the hazard of fire. Additionally, the fire event itself is taken to be instantaneous and therefore, only prevention management spending that occurs exactly at the time of fire will decrease the number of acres burned in the fire. Any prevention management spending before the time of fire does nothing to decrease the number of acres destroyed in the fire.

Consider a forest with $\bar{A}$ acres over the finite time horizon $[0, T]$. Let $A(t)$ be the number of unburned acres in a forest before a fire at time $\tau \in [0, T]$. Suppose the forest generates a flow of non-timber benefits B per unit time as a function of the number of unburned acres in the forest; that is, $B = B\big(A(t)\big)$. Non-timber benefits are the sum of all provisioning, regulatory, supporting, and cultural ecosystem services provided by the forest. The focus on non-timber benefits is consistent with forests where fuel management is costly, but may not fully capture fuel management incentives associated with service contracts. For now, suppose the next large fire in the forest occurs at time τ with $0 < \tau < T$. Before a fire at time τ the present value of the net benefit from the forest is given by

$$\int_0^\tau \Big[B\big(A(t)\big) - h(t)\Big]e^{-rt}dt, \tag{1}$$

where $h(t)$ is the prevention management spending rate over time. The number of unburned acres $A(t)$ before the fire is governed by the differential equation

$$A'(t) = \delta\big(\bar{A} - A(t)\big) \text{ with } A(0) = A_0 \leq \bar{A}, \tag{2}$$

where δ represents the regeneration rate of the forest. We assume that the regeneration of the forest is only dependent on the number of unburned acres in the forest and the initial condition A_0; it is not dependent on any control variables. The solution of the differential equation (2) for the number of unburned acres is

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}. \tag{3}$$

We formulate our optimal control problem to allow for time-varying unburned acres before a fire. This will later enable us to apply the optimal control problem successively in order to consider a sequence of fires.

2.1 *Ex Post* Fire Suppression

The number of acres destroyed in the fire, K, is dependent on the *ex ante* prevention management expenditures at the time of the fire, $h(\tau)$, and the *ex post* fire suppression expenditures at the time of the fire, $x(\tau)$. That is,

$$K = K\big(h(\tau), x(\tau)\big). \tag{4}$$

Assume that the number of acres burned in the fire K is decreasing with respect to increases in prevention management and suppression spending; i.e. $\frac{\partial K}{\partial h} < 0$ and $\frac{\partial K}{\partial x} < 0$.

Let $\hat{A}(t)$ represent the number of unburned acres in the forest following a fire at time τ. The fire event at time τ is taken to be instantaneous and so the number of unburned acres destroyed in the fire K is taken into account at the time of fire τ. Thus, the number of unburned acres at the time of fire τ, $\hat{A}(\tau)$, represents the number of acres remaining in the forest after the number of acres destroyed K in the fire have been accounted for:

$$\hat{A}(\tau) = A(\tau) - K\big(h(\tau), x(\tau)\big). \tag{5}$$

At the time of fire there is a jump discontinuity between $A(\tau)$ and $\hat{A}(\tau)$. Following previous work, for the optimal control problem formulation we assume that another fire does not occur in our finite time horizon $[0, T]$. We will be considering a sequence of fires in Section 4. We assume that starting from the time of fire τ the number of unburned acres $\hat{A}$ in the forest increases according to the differential equation

$$\hat{A}'(t) = \delta\big(\bar{A} - \hat{A}(t)\big) \text{ with } \hat{A}(\tau) = A(\tau) - K\big(h(\tau), x(\tau)\big), \tag{6}$$

so that $\hat{A}(t)$ increases toward $\bar{A}$ as time increases. Note that $A(\tau)$ is known from equation (3). The solution to this differential equation is

$$\hat{A}(t) = \bar{A} - \Big(\bar{A} - \Big(A(\tau) - K\big(h(\tau), x(\tau)\big)\Big)\Big)e^{-\delta(t-\tau)}. \tag{7}$$

As before, the fire event is taken to be instantaneous and so are the associated costs. The jump discontinuity in non-timber benefits, due to the jump discontinuity in the number of unburned acres in the forest, serves as a cost of the fire. Additionally, the cost of suppressing the fire $x(\tau)$ and the cost of damages to built structures D are subtracted from the non-timber benefits that accrue after the fire. The damages to built structures is a function of the number of acres destroyed in the fire:

$$D = D\Big(K\big(h(\tau), x(\tau)\big)\Big). \tag{8}$$

This may include impacts to surrounding buildings, roads, etc. We assume that larger fires are more likely to impact built structures: $\frac{\partial D}{\partial K} > 0$. Additionally, we assume that built structures could be saved by increasing prevention and suppression spending: $\frac{\partial D}{\partial h} < 0$ and $\frac{\partial D}{\partial x} < 0$.

The function describing the flow of benefits before and after the fire is the same, even though we distinguish between unburned acres before the fire and unburned acres after the fire, A and $\hat{A}$, respectively. The net present value of the forest following a fire is given by the non-timber benefits accrued from the time of fire to the end of our time horizon net of the instantaneous suppression costs and costs to built structures:

$$\int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-rt}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big]e^{-r\tau}, \tag{9}$$

subject to (7) and $x(\tau) \geq 0$. With only a single fire event, there is no incentive to invest in prevention following a fire. When we move to consider sequences of fires we will have prevention management following each fire event, but this is because we are essentially "resetting" our optimal control problem after every fire.

Let the value of the forest after the fire, with $e^{-r\tau}$ factored out, be defined by

$$JW\big(\tau, A(\tau), h(\tau), x(\tau)\big) = \int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-r(t-\tau)}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big]. \tag{10}$$

Note that the *ex post* value of the forest is a function of the time of fire τ, the prevention management spending $h(\tau)$, suppression spending $x(\tau)$, and the number of unburned acres $A(\tau)$ at the time of fire, before the effects of the fire have been considered. We say that JW is a function of $A(\tau)$ and not $\hat{A}(\tau)$ because $\hat{A}$ is determined by the boundary condition containing $A(\tau)$ and the differential equation (6). Hence, given a time of fire τ, the corresponding prevention management spending at that time $h(\tau)$, and the number of unburned acres $A(\tau)$ before the effects of the fire have been considered, the optimal *ex post* value of the forest is the solution to

$$\sup_{x(\tau)} \int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-r(t-\tau)}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big]$$

$$\text{subject to } x(\tau) \geq 0, \tag{11}$$

$$\text{where } \hat{A}(t) = \bar{A} - \bigg(\bar{A} - \Big(A(\tau) - K\big(h(\tau), x(\tau)\big)\Big)\bigg)e^{-\delta(t-\tau)}, \tag{12}$$

with $x(\tau)$ being a real-valued scalar representing suppression spending. Let $x^*(\tau)$ be the real-value scalar representing optimal suppression spending for a given τ, $h(\tau)$, and $A(\tau)$, which maximizes the value of the forest after the fire. The maximized *ex post* value of the forest for a given τ, $h(\tau)$, and $A(\tau)$ is henceforth denoted by

$$JW^*\big(\tau, A(\tau), h(\tau)\big) = JW\big(\tau, A(\tau), h(\tau), x^*(\tau)\big). \tag{13}$$

The value of the forest following a fire JW is maximized when evaluated at $x^*(\tau)$. We assume that suppression spending increases the value of the forest following a fire:

$$\frac{\partial JW^*\big(\tau, A(\tau), h(\tau)\big)}{\partial h} > 0. \tag{14}$$

Once functional forms are chosen we explicitly determine $x^*(\tau)$, and thus $JW^*\big(\tau, A(\tau), h(\tau)\big)$, using scalar optimization techniques. The details surrounding this process are discussed in Section 3.

2.2 ***Ex Ante*** Fire Prevention

If the time of fire $\tau \in [0, T]$ is strictly less than T, then the total value of the forest over the time horizon $[0, T]$ is given by the sum of the net value of the forest before the fire and the net value of the forest after the fire up to time T,

$$\int_0^\tau \big[B\big(A(t)\big) - h(t)\big] e^{-rt} dt + \int_\tau^T B\big(\hat{A}(t)\big) e^{-rt} dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big] e^{-r\tau}, \tag{15}$$

where $A(t)$ is given by (3) and $\hat{A}(t)$ is given by (7). Note that this total value is the sum of (1) and (10) and gives the value of the forest over the full time horizon $[0, T]$.

If the time of the first fire τ is equal to T, then we represent the value of the forest over the time horizon $[0, T]$ by

$$\int_0^T \Big[B\big(A(t)\big) - h(t)\Big] e^{-rt} dt, \tag{16}$$

where $A(t)$ is given by (3). In this case, we recognize that a fire will eventually occur, but because it does not occur within the time horizon $[0, T]$ we do not subtract the instantaneous suppression costs or cost of damages to built structures.

In summary, the value of the forest over $[0, T]$ depends on the time of fire τ, the prevention management spending h, and the initial condition $A_0 = A(0)$ for the number of unburned acres in the forest before a fire. The value of the forest can thus be represented by the piecewise function

$$\mathcal{V}(A_0, \tau, h) = \begin{cases} \int_0^{\tau} \Big[B\big(A(t)\big) - h(t)\Big] e^{-rt} dt + e^{-r\tau} JW^*\big(\tau, A(\tau), h(\tau)\big) & \text{if } \tau < T \\ \int_0^{T} \Big[B\big(A(t)\big) - h(t)\Big] e^{-rt} dt & \text{if } \tau = T, \end{cases} \tag{17}$$

where $A(t)$ is given by (3). Note that $\hat{A}$ is completely contained within JW^*. The equation $\mathcal{V}(A_0, \tau, h)$ represents the net present value of the forest over the whole time interval $[0, T]$ for a given time of fire τ, prevention management spending h, and initial number of unburned acres in the forest A_0. In the case that a fire happens within the time horizon, $\mathcal{V}$ incorporates the optimal value of the forest following a fire $JW^*\big(\tau, A(\tau), h(\tau)\big)$.

When the large fire event will occur is unknown. Thus, the time of fire $\tau \in [0, T]$ represents an uncertainty in our system. To capture this uncertainty in our problem, we take the time of fire τ to be a realization of the mixed-type random variable (RV) $\mathcal{T}$. The random variable is characterized by the hazard function ψ, defined as

$$\psi = \lim_{\Delta t \to 0} \left\{ \frac{Pr(\text{fire in } [t, t + \Delta t) | \text{no fire up to } t)}{\Delta t} \right\}. \tag{18}$$

The hazard function represents the conditional probability that a fire will occur at a time t given that no fire has occurred up to that time. For our problem, the hazard function is assumed to be a function of the *ex ante* prevention management spending rate,

$$\psi = \psi\big(h(t)\big). \tag{19}$$

Furthermore, we assume that the hazard is decreasing with respect to an increased prevention management spending rate, i.e. $\frac{\partial \psi}{\partial h} < 0$. A constant background hazard is assumed in the absence of *ex ante* prevention management spending.

The survivor function $S(t)$, which gives the probability of the forest surviving to time t with no fire, is related to the hazard function ψ in the following way:

$$S(t) = e^{-\int_0^t \psi\left(h(z)\right)dz}. \tag{20}$$

It follows that $S(0) = 1$. While we assume that prevention spending can reduce hazard, we do not assume that prevention spending will indefinitely delay the occurrence of a large, stand-replacing fire. Therefore, we assume that the integral representing the cumulative hazard, $\int_0^t \psi\big(h(z)\big)dz$, will diverge to positive ∞ as $t \to \infty$, and thus $S(\infty) = 0$. The corresponding cumulative distribution function for $\mathcal{T}$ is related to the survivor function and is given by

$$F_{\mathcal{T}}(\tau) = \begin{cases} 1 - S(\tau) & \text{if } \tau < T \\ 1 & \text{if } \tau = T. \end{cases} \tag{21}$$

Notice the potential for discontinuity at time T. Hence, we observe that the probability density function for $\mathcal{T} \in [0, T)$ is

$$f_{\mathcal{T}}(t) = \psi\big(h(t)\big)S(t). \tag{22}$$

The mixed type RV $\mathcal{T}$ has a discrete component. For $\mathcal{T} = T$, the probability mass is

$$\begin{aligned} P(\mathcal{T} = T) &= F_{\mathcal{T}}(T) - F_{\mathcal{T}}(T^-) \\ &= 1 - \big(1 - S(T)\big) \\ &= S(T). \end{aligned} \tag{23}$$

Again, if $\tau = T$, no costs other than prevention management spending h are considered. Our goal is to determine the prevention management spending rate $h(t) \geq 0$ which maximizes the net present value of the forest over $[0, T]$ using deterministic optimal control. As written, our problem is currently stochastic. However, using techniques developed by Reed, we can convert this stochastic problem to deterministic by taking the expectation of (17) with respect to the RV $\mathcal{T}$ and introducing a state variable to represent cumulative hazard [31].

The expected net present value of the forest over $[0, T]$, is given by

$$\begin{aligned} J(h) &= E_{\mathcal{T}}\big\{\mathcal{V}(A_0, \tau, h)\big\} \\ &= \int_0^T \left[\int_0^\tau \Big[B\big(A(t)\big) - h(t)\Big]e^{-rt}dt + JW^*\big(\tau, A(\tau), h(\tau)\big)e^{-r\tau}\right]\psi\big(h(\tau)\big)S(\tau)d\tau \\ &\quad + S(T)\int_0^T \Big[B\big(A(t)\big) - h(t)\Big]e^{-rt}dt. \end{aligned} \tag{24}$$

After a bit of calculus, we arrive at

$$J(h) = \int_0^T \Big[B\big(A(t)\big) - h(t) + \psi\big(h(t)\big)JW^*\big(t, A(t), h(t)\big)\Big]S(t)e^{-rt}dt. \tag{25}$$

This function, $J(h)$, represents the expected net present value of the forest over an interval $[0, T]$ subject to the survivor function $S(t)$. By introducing a new state variable y to represent cumulative hazard we complete the conversion of our stochastic problem to deterministic. Let y represent cumulative hazard and be governed by the differential equation

$$y'(t) = \psi\big(h(t)\big) \text{ with } y(0) = 0. \tag{26}$$

The initial condition $y(0) = 0$ follows from the fact that $S(0) = 1$. Note that the survivor function can be rewritten as

$$S(t) = e^{-y(t)}, \tag{27}$$

and this allows us to rewrite (25) with our new state variable y.

Our goal is to find a control h in our control set which maximizes the objective functional $J(h)$ with respect to the state variable y governed by differential equation (26). Therefore, our deterministic optimal fire prevention problem can be written as

$$\sup_{h\in U} \int_0^T \Big[B\big(A(t)\big) - h(t) + \psi\big(h(t)\big)JW^*\big(t, A(t), h(t)\big)\Big]e^{-rt-y(t)}dt \tag{28}$$

$$\text{subject to } y'(t) = \psi\big(h(t)\big) \text{ with } y(0) = 0, \tag{29}$$

where

$$U = \big\{h : [0, T] \to [0, \infty) | h \text{ is Lebesgue measurable}\big\}, \tag{30}$$

and

$$A(t) = \bar{A} - (\bar{A} - A_0)e^{-\delta t}. \tag{31}$$

Thus, our control problem with stochastic time of fire has been converted to a deterministic optimal control problem.

2.3 Linking Optimal Prevention and Suppression

Selecting explicit functional forms for B, K, D, and the hazard function allows us to determine the the optimal *ex post* value of the forest following a fire $JW^*\big(\tau, A(\tau), h(\tau)\big)$ and ultimately solve our optimal control problem by determining the optimal management spending $h(t)$ rate over $[0, T]$. The benefits function B represents the flow of benefits from the forest and is assumed directly proportional to the number of unburned acres in the forest:

$$B\big(A(t)\big) = B_1 A(t), \tag{32}$$

where parameter $B_1 \geq 0$. The number of acres completely burned by the fire, K, is decreasing with respect to prevention management and suppression expenditures:

$$K(h, x) = \frac{k}{(k_1 + h)(k_2 + x)}, \tag{33}$$

with parameters $k > 0$ and $k_1, k_2 \geq 1$. The parameter k is related to the size of a fire. The parameter k_1 controls the magnitude of the effect of prevention management spending h on decreasing the number of acres burned. Similarly, the parameter k_2 controls the magnitude of the effect of suppression spending x on decreasing the number of acres burned. It is assumed that the cost of lost built structures is directly proportional to the number of acres destroyed in the fire:

$$D\big(K(h,x)\big) = cK(h,x) = \frac{ck}{(k_1+h)(k_2+x)}, \tag{34}$$

with parameter $c \geq 0$ as the cost of damages in millions of dollars per thousand acres burned.

The hazard function ψ, representing the conditional probability that a fire will occur at time t given that a fire has not occurred up to that time:

$$\psi\big(h(t)\big) = be^{-vh(t)}, \tag{35}$$

is consistent with the literature [31, 26, 4, 7]. The parameter $0 < b < 1$ represents the constant hazard rate when there is no prevention management spending. The constant $v > 0$ is used to control the effectiveness of preventative management spending $h(t)$ on reducing hazard.

Now that the functional forms have been defined, we optimize the value of the forest after the fire JW. Recall the *ex post* problem is to maximize

$$\max_{x(\tau)} \int_{\tau}^{T} B\big(\hat{A}(t)\big)e^{-r(t-\tau)}dt - \Big[D\Big(K\big(h(\tau), x(\tau)\big)\Big) + x(\tau)\Big] \tag{36}$$

subject to $x(\tau) \geq 0$,

where $\hat{A}(t) = \bar{A} - \Big(\bar{A} - \Big(A(\tau) - K\big(h(\tau), x(\tau)\big)\Big)\Big)e^{-\delta(t-\tau)}$. (37)

Using the solution to the state differential equation for $\hat{A}(t)$ above, we integrate the flow of benefits from the time of fire τ to the end of our time horizon T. Hence, the *ex post* value of the forest is given by

$$
\begin{aligned}
&JW\big(\tau, A(\tau), h(\tau), x(\tau)\big) \\
&\quad= \frac{B_1 \bar{A}}{r}\left(1 - e^{-r(T-\tau)}\right) - \frac{B_1\big(\bar{A} - A(\tau)\big)}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) \\
&\qquad - K\big(h(\tau), x(\tau)\big)\left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right] - x(\tau).
\end{aligned} \tag{38}
$$

Our goal is to maximize $JW\big(\tau, A(\tau), h(\tau), x(\tau)\big)$ with respect to the one-time suppression costs $x(\tau)$. We do this using scalar optimization and thus consider the partial derivative of JW (38) with respect to $x(\tau)$. It follows that

$$
\begin{cases}
x^*(\tau) = 0 & \text{if } \frac{\partial JW}{\partial x(\tau)} < 0, \\
x^*(\tau) \geq 0 & \text{if } \frac{\partial JW}{\partial x(\tau)} = 0.
\end{cases} \tag{39}
$$

As K is a function of x, the partial derivative of $JW\big(\tau, A(\tau), h(\tau), x(\tau)\big)$ with respect to $x(\tau)$ is,

$$
\begin{aligned}
\frac{\partial JW}{\partial x} &= -\left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right]\frac{\partial K}{\partial x} - 1 \\
&= \left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right]\frac{k}{(k_1 + h)(k_2 + x)^2} - 1.
\end{aligned} \tag{40}
$$

If $\frac{\partial JW}{\partial x(\tau)} = 0$, then $x^*(\tau) \geq 0$. To determine $x^*(\tau)$ in this case we set the partial derivative (40) equal to zero, and solve for $x(\tau)$. As $x(\tau) \geq 0$, we determine

$$
0 \leq x^*(\tau) = x^*\big(\tau, h(\tau)\big) = \sqrt{\frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\left(1 - e^{-(\delta+r)(T-\tau)}\right) + c\right]} - k_2, \tag{41}
$$

in the case that $\frac{\partial JW}{\partial x(\tau)} = 0$.

If $\frac{\partial JW}{\partial x(\tau)} < 0$ and $x^*(\tau) = 0$, then

$$\sqrt{\frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right]} - k_2 < 0 = x^*(\tau). \tag{42}$$

Based on our choices for our functional forms, we see that optimal suppression spending x^* is not only a function of the time of fire, but also prevention management spending $h(\tau)$. Note that maximum of JW could occur at the endpoint. Thus, it follows that optimal suppression spending is given by

$$x^*\big(\tau, h(\tau)\big) = \max\left\{0, \sqrt{\frac{k}{\big(k_1 + h(\tau)\big)}\left[\frac{B_1}{\delta + r}\Big(1 - e^{-(\delta+r)(T-\tau)}\Big) + c\right]} - k_2\right\}. \tag{43}$$

Therefore, we substitute (43) into our JW representation (38) so that the optimal value of the forest following a fire is

$$JW^*\big(\tau, A(\tau), h(\tau)\big) = JW\Big(\tau, A(\tau), h(\tau), x^*\big(\tau, h(\tau)\big)\Big). \tag{44}$$

We note that a quick calculation shows $\frac{\partial^2 JW}{\partial x^2} \leq 0$ and so the JW value (41) is indeed a maximum of JW (38).

We now work through the derivation of the conditional current-value optimality system. Let the standard Hamiltonian be given by H and let the adjoint function associated with state variable y by given by λ. Let the conditional current-value adjoint function be given by

$$\rho(t) = e^{rt+y(t)}\lambda(t). \tag{45}$$

The conditional current-value Hamiltonian $\mathcal{H}$ is

$$\mathcal{H} = e^{rt+y(t)} H \tag{46}$$

$$= B\big(A(t)\big) - h(t) + \psi\big(h(t)\big) JW^*\big(t, A(t), h(t)\big) + \rho(t)\psi\big(h(t)\big). \tag{47}$$

The partial derivative of the conditional current value Hamiltonian $\mathcal{H}$ with respect to the control is

$$\frac{\partial \mathcal{H}}{\partial h} = -1 + JW^*\big(t, A(t), h(t)\big)\frac{\partial \psi}{\partial h} + \frac{\partial JW^*}{\partial h}\psi\big(h(t)\big) + \rho(t)\frac{\partial \psi}{\partial h}. \tag{48}$$

Special care must be taken when deriving the conditional current-value adjoint differential equation. Since $\rho(t) = e^{rt+y(t)}\lambda(t)$, the differential equation for the conditional current value adjoint is given by

$$\rho'(t) = \Big(r + \psi\big(h(t)\big)\Big) e^{rt+y(t)}\lambda(t) + e^{rt+y(t)}\lambda'(t) \tag{49}$$

$$= \Big(r + \psi\big(h(t)\big)\Big)\rho(t) - e^{rt+y(t)}\frac{\partial H}{\partial y} \tag{50}$$

Hence, the conditional current-value adjoint differential equation is

$$\rho'(t) = \Big(r + \psi\big(h(t)\big)\Big)\rho(t) + B\big(A(t)\big) - h(t) + \psi\big(h(t)\big) JW^*\big(t, A(t), h(t)\big), \tag{51}$$

with transversality condition

$$\rho(T) = e^{rT+y(T)}\lambda(T) = 0. \tag{52}$$

The hazard function ψ is nonlinear in h, as is the function $JW^*\big(t, A(t), h(t)\big)$, which represents the optimal value of the forest following a forest fire for a given

Table 1: The table below includes the parameter values chosen to reflect the 2011 Las Conchas Fire.

Parameter	Units	Value	Justification
$\bar{A}$	acres(1000)	1700	size of SFNF, BNM, VCNP
r	/time	0.04	standard discount rate
k	acres(1000)×\$^{2}/time	7000	$k \approx$ size of fire × suppression \$
k_1	\$ (mil.)/time	1	assumed
k_2	\$ (mil.)/time	1	assumed
δ	/time	0.05	Pipo: 70-250 years to mature
b	———	0.2	high frequency of fires in region
c	\$ (mil.)/ acres(1000)	0.1	114 buildings destroyed, 156,000 acres burned
B_1	\$ (mil.)/time	0.02	calculated from x^* formula
v	———	1	assumed

time of fire and a given amount of prevention management spending at the time of fire. We utilize the fact that Pontryagin's Maximum Principle (PMP) states that the optimal control maximizes the Hamiltonian with respect to the control h pointwise at each t to numerically determine the optimal control [15]. We justify the use of PMP for our maximization problem as we can show that

$$\frac{\partial^2 \mathcal{H}}{\partial h^2} \leq 0, \tag{53}$$

numerically for our given functions and parameter choices.

3 Las Conchas Fire: Numerical Example

Now that we have formulated our optimal control problem and the associated optimality system, we solve it numerically and interpret the results. To help us build a realistic set of parameter values, we examine information from the 2011 Las Conchas Fire . A fallen power line ignited the Las Conchas Fire on June 26,

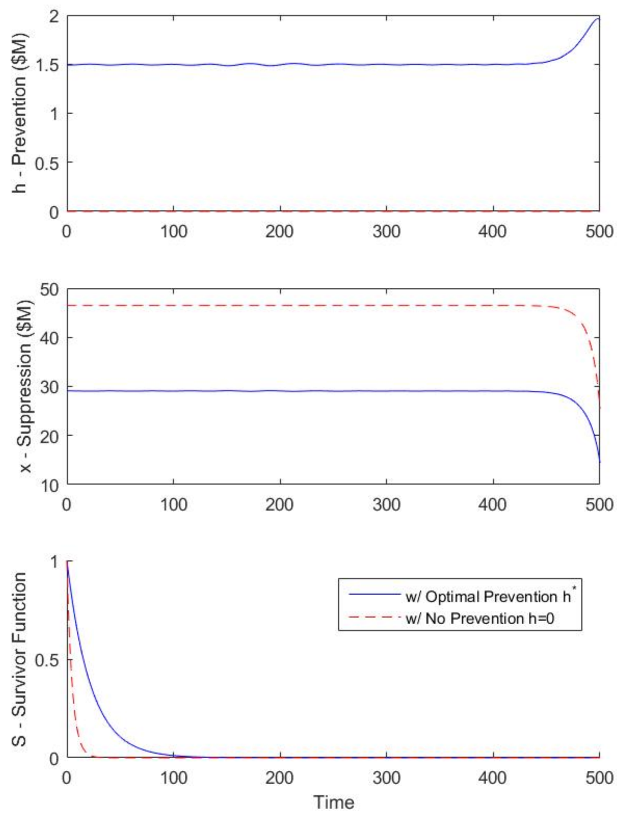

Figure 1: The plots above contain the h^*, x^*, and S results of our optimal control problem using the Las Conchas Fire parameter set. For comparison, in each plot we include the case with optimal prevention management spending h^* and the case with no prevention spending $h = 0$.

2011. The Las Conchas Fire continued to burn over the course of the summer through sections of Santa Fe National Forest, Bandelier National Monument, and Valles Caldera National Preserve near Los Alamos, New Mexico. The fire was finally contained at the beginning of August 2011 [24, 36]. Over 150,000 acres burned in the fire and over \$40 million were spent on fire suppression efforts [24, 36, 42]. In addition to suppression costs, over 110 structures were destroyed or damaged during the fire [24, 36]. We are using the data from this fire to build a more realistic problem, not to draw any retrospective conclusions concerning prevention management or suppression spending decisions made at the time of this fire. Parameter choices are summarized in Table 1.

The parameter $\bar{A}$ represents the "size of the forest" in units of thousands of acres. The Las Conchas Fire mainly burned through the Santa Fe National Forest, Bandelier National Monument, and Valles Caldera National Preserve and the combined size of these three areas is approximately 1,722 thousand acres[23, 41, 44]. Rounding down, we take $\bar{A} = 1,700$.

The parameter δ represents the regeneration rate of the forest following a fire. We choose δ based on the dominant tree type in the forest, which in the Santa Fe National Forest is Ponderosa Pine (Pipo). The age of Ponderosa Pine at maturity is 70-250 years [40]. Assuming that at the time of fire the number of unburned acres is reduced by half, we choose a value for δ so that the number of unburned acres after 100 years has approximately returned to $\bar{A}$. As such, we choose $\delta = 0.05$. The discount rate is set at 4 percent in accordance with USDA Forest Service practice: $r = 0.04$ [32]. The parameter b, found in the hazard function $\psi\big(h(t)\big)$, represents the background fire hazard. To capture the probability that large, high-severity fires happen frequently in the region, we choose $b = 0.2$ [45]. For $b = 0.2$ the probability of the forest surviving to 3.5 years with no fire is approximately 0.5.

The parameters $k, k_1,$ and k_2 are all included in the function K which represents the number of acres burned in the instantaneous fire. Given that values for parameters k_1 and k_2 could not be estimated from the literature, for simplicity we choose $k_1 = k_2 = 1$. The parameter k, with $k_1 = k_2 = 1$, represents the number of acres that will be completely burned in the fire given that there is no prevention management spending or suppression spending at the time of the fire. In particular, given that we know the number of acres burned in the Las Conchas Fire and the amount spent on suppression, we use the function K to estimate a value for the fire severity parameter k. Assuming that there is no prevention management spending at the time of fire, we have $k \approx K \times (1 + x)$, where K represents the number of acres burned (in thousands) in the fire and x represents the amount spent on suppression (in millions). For the Las Conchas Fire, approximately 157 thousand acres were burned and over \$40M were spent on suppression. Thus, we choose $k = 7,000$ given that estimates for suppression costs range between \$40M and \$50M.

The parameter v is found in the hazard function ψ and represents the effectiveness of prevention management spending on reducing hazard. From the literature, we choose $v = 1$ [27, 4].

The parameter c represents the cost of damages to built structures in millions of dollars per thousand acres burned. In the Las Conchas Fire, 114 buildings were destroyed or damaged in the fire [36]. The median value of homes in the region ranges between \$100,000 and \$450,000 [39]. First, we estimate the cost of non-timber damages D for the Las Conchas Fire by multiplying the number of building destroyed by the median value of homes in the area. Our estimate for D is thus \$17.2 million. We then divide this estimate for D by the number of acres destroyed in the fire, $K = 157$, to estimate an appropriate value for c; $c \approx \frac{D}{K}$. For the Las Conchas Fire in particular, we round and take $c = 0.1$.

Lastly, we look to determine an appropriate value for the parameter B_1, which represents the flow of non-timber benefits from the forest given in units millions of dollars per thousand acres. The challenge of valuing a forest is very complex and is its own problem in and of itself [25]. We use our equation for optimal suppression spending (41) to determine a value for B_1 based on our other parameter choices and the amount of money spent on suppression for the Las Conchas Fire. In order to determine a reasonable estimate for B_1, we assume that the amount of suppression spending was optimal and we allow this amount to stand in for x^*. We then solve equation (41) for B_1 and approximate its value using our previous parameter choices. Furthermore, we assume $h(\tau) = 0, \tau = 0,$ and $T = 500$. This leads to the choice of $B_1 = 0.02$ for the Las Conchas Fire.

We recognize that the selection of some of these parameter values is not literature driven. Because of this, we perform a sensitivity analysis to determine which parameters have the most impact on the overall expected net present value of the forest. We use Latin Hypercube Sampling (LHS) and Partial Rank Correlation Coefficient (PRCC) analysis to determine the parameters to which the value of the objective functional evaluated at the optimal control h^* is most sensitive. Ten parameters with appropriate ranges were chosen for this analysis and the full details are shown in the appendix. We conclude that the parameter B_1 has the strongest impact on the expected net present value of the forest $J(h^*)$, followed by parameters $\bar{A}$ and r, followed by parameters c and k_2. We note that using the objective functional at the optimal control as an output for a LHS/PRCC analysis is novel.

We consider a time horizon of $T = 500$ years. We choose a long time horizon so that any tail effects from the finite time horizon can be reasonably ignored. Additionally, by choosing T very large, we essentially “guarantee” that the time of the next fire will fall within our time horizon. This is validated by the survivor functions in Figure 1 which is essentially zero after 100 years in both the optimal prevention management spending case and the no prevention management spending case. This long time horizon is also very important in our consideration of sequences of fires.

From our numerical results in Figure 1, the optimal prevention management spending rate h^* is approximately constant at 1.5 million dollars per year over the course of the time horizon, with an increase to 2 million near the end of the time horizon. This increase is likely due to the sharp decrease in the value of JW^* near the end of the time horizon. Hence, we interpret the graph of h^* as saying that approximately \$1.5 million should be spent on prevention management per year, up to the time of the first fire, which in practice is unknown.

Recall that the fire event in our problem is taken to be instantaneous, along with its associated costs. As seen in Figure 1, the function representing optimal suppression spending in the optimal prevention management spending case is approximately constant at \$29M over the time horizon, except for effects at the end. It is important to recall that suppression spending is a one-time instantaneous cost at the time of the fire and therefore, the suppression cost of \$29M only occurs once in application at the time of fire. This is in contrast to the optimal prevention management spending h^*, discussed in the previous paragraph, which is ongoing up to the time of fire. In the case without prevention management spending, instantaneous suppression spending is roughly \$46.5M at the time of fire, given that the function for x^* in the case without prevention management spending is approximately constant over the course of the time horizon. As expected, instantaneous suppression spending in the case without prevention management spending is greater than instantaneous suppression spending in the optimal prevention case: $\frac{\partial x^*}{\partial h} < 0$. Moreover, instantaneous suppression spending decreases approximately 38% when optimal prevention management spending is applied, in comparison to the corresponding amount without prevention management spending.

Given that the time of fire is treated as a random variable in our problem, we would like to compare the expected time of the next fire between the two cases of optimal prevention management and no prevention management. In order to determine this value, we calculate the expected value of the time of fire random variable. The expected value of the time of fire random variable is justifiably approximated by

$$E[\mathcal{T}] = \int_0^T t\psi\big(h(t)\big)e^{-y(t)}dt. \tag{54}$$

In the case of no prevention management spending, this is reduced to

$$\int_0^T bte^{-bt}dt. \tag{55}$$

The mean time of fire in the case without prevention management spending is 5 years. In the case of optimal prevention management spending, the mean time of fire is approximately 22.3 years. Hence, on average, the time of the fire in the optimal case is approximately 17 years later than the no prevention case. Furthermore, in the case of multiple fires, we might expect that over a fixed amount of time there will be fewer fires when optimal prevention management spending is employed compared to when there is no prevention management spending.

Another measure we wish to consider is the expected net present value of the forest over $[0, T]$. This is given by the value of the objective functional in our optimal control problem, either evaluated at the optimal control h^* or evaluated when $h = 0$. In the no prevention management spending case, the expected value of the forest over $[0, T]$ is approximately \$772.6M. In the optimal prevention spending case, the expected value of the forest over $[0, T]$ is approximately \$801.2M. Thus, the value of forest is larger when optimal prevention management spending is applied.

Overall, we see that in the case of optimal prevention management spending h^*, the value of the forest, and the mean time of fire are larger than in the case without prevention management spending, $h = 0$. However, we recognize that it is unrealistic to assume that only one fire will occur in 500 years, especially since we chose the background hazard b to reflect a high frequency of fires in the region. Thus, in order to make better comparisons concerning the value of the forest and the trade-offs between prevention management and suppression spending, we would like to apply our optimal control problem to a sequence of fires over a fixed amount of time in Section 4.

4 Applying Optimal Prevention Strategies to a Sequence of Fires

Our goal is to explore the effects of prevention management spending on the value of a forest over a fixed number of years given that a sequence of an unknown number of large fire events may occur within this time. Let this fixed management horizon that we wish to consider a sequence of fires over be Y years long.

We are optimizing prevention management spending between each fire event using our optimal control problem. We determine J_Y, the value of the forest over Y years, and consider the trade-offs in total prevention management spending and suppression spending. Because we are sampling the times of the fires, each time we determine J_Y years will be different. Thus, we perform a simulation study and perform multiple trials. We then examine some basic descriptive statistics for the value of the forest over Y years, the number of fires over Y years, and the total amount of prevention management spending and suppression spending over Y years. For comparison, we also consider the case without prevention management spending.

4.1 Fire Sequence Simulation

As our optimal control problem allows for non-constant unburned acres before a fire, it possible to consider sequences of fires. In essence, we solve our optimal control problem, use y^* to build the CDF of RV $\mathcal{T}$, sample for a time of fire, and then solve our optimal control problem again with an updated initial condition A_0 for the number of unburned acres in the forest. This new initial condition takes into account the number of acres destroyed in the fire according to the previous solution of the optimal control problem. We continue to do this until the time of the n^{th} fire, n unknown, is beyond a specified amount of time, Y.

Now we discuss some important differences between the quantities Y and T. The parameter T represents the length of the time horizon considered for our optimal control problem. The parameter Y represents the the length of the management horizon over which we want to consider a sequence of fires. Over the course of the management horizon $[0, Y]$ our optimal control problem will be solved several times. Each time the optimal control problem is solved over the time horizon $[0, T]$. We choose Y based on the number of years over which we want to consider a sequence of fires. The length of the time horizon for our optimal control problem T should be chosen so that $S(T)$ is very small (close to zero) so that we can approximate the CDF for $\mathcal{T}$ by its continuous counterpart.

Here, we explain the process used for a single simulation of a sequence of fires. Within a single simulation, or trial, we solve our optimal control problem multiple times and as such we will need to distinguish between the different state and control variables corresponding to the different solutions for the optimal control problem. To do this, we use numerical subscripts to indicate which solution to which the variables correspond.

First, we solve our optimal control problem for a given set of parameters and initial condition $A_1(0) = \bar{A}$. As a result, we know the optimal prevention management spending $h_1^*(t)$, the optimal instantaneous suppression spending $x_1^*(t)$, the optimal cumulative hazard $y_1^*(t)$, and the number of unburned acres $A_1(t)$ over the time horizon $[0, T]$. Note the subscript 1 on the variables denotes that these functions correspond to the solution of our optimal control problem the first time we solve it. The number of unburned acres $A_1(t)$ is unaffected by either control variables x and h and as such, we do not use the star notation with it; it is completely determined by (3). After numerically determining the solution, we build the CDF for the time of fire RV $\mathcal{T}$ using y^* and sample for the time of the first fire.

Let $\tau_1 \in [0, T]$ be the sampled time of the first fire. If $\tau_1 > Y$, then the value of the forest, denoted by J_Y, over Y years is given by

$$J_Y = \int_0^Y \Big[B\big(A_1(t)\big) - h_1^*(t)\Big] e^{-rt} dt, \tag{56}$$

We do not consider costs of suppression or non-timber damages because the time of the first fire is outside of our management horizon Y. If $\tau_1 = Y$, then the value of the forest up to time $\tau_1 = Y$ is given by

$$J_Y = \int_0^Y \Big[B\big(A_1(t)\big) - h_1^*(t)\Big] e^{-rt} dt - \Big[D\Big(K\big(h_1^*(\tau_1), x_1^*(\tau_1)\big)\Big) + x_1^*(\tau_1)\Big] e^{-r\tau_1}, \tag{57}$$

Here, we consider the cost of suppression and cost to built structures because the time of the first fire occurs at the end of the management horizon Y. If $\tau_1 < Y$, then value of the forest up to time τ_1 is given by

$$\int_0^{\tau_1} \Big[B\big(A_1(t)\big) - h_1^*(t)\Big]e^{-rt}dt - \Big[D\Big(K\big(h_1^*(\tau_1), x_1^*(\tau_1)\big)\Big) + x_1^*(\tau_1)\Big]e^{-r\tau_1}, \quad (58)$$

and we need to solve our optimal control problem again and sample for the time of the next fire since $\tau_1 < Y$. The expression directly above is not labeled as J_Y because we have not yet accounted for the whole management horizon.

Now that a fire has occurred, and $\tau_1 < Y$, the number of unburned acres is less than $\bar{A}$ and we need to set the initial condition $A_2(0)$ to prepare for the next application of our optimal control problem. In particular, we set our new initial condition to be

$$A_2(0) = A_1(\tau) - K\big(h_1^*(\tau_1), x_1^*(\tau_1)\big), \quad (59)$$

where $A_1(\tau) = \bar{A}$. Note that $A_1(\tau) = \bar{A}$ because for our first solution of our optimal control problem we chose the initial condition for A to be at an equilibrium point. We point out again that while this initial condition is dependent on prevention management spending and suppression spending, it is from the previous optimal control solution, and thus completely known.

Note that while we are considering these fires in sequence, the time horizon $[0, T]$ of our optimal control problem remains the same. At each fire event, we are in essence "resetting" our problem. With our new initial condition, we solve our optimal control problem using the same set of parameters over $[0, T]$ and once again, as a result, we will know $h_2^*(t)$, $x_2^*(t)$, and $y_2^*(t)$ over the time horizon $[0, T]$. Thus, as before, we sample for the time of the second fire, τ_2, using the

CDF constructed using y_2^*. The sampled time τ_2 is associated with the time horizon $[0,T]$. We have to be careful when translating this to our management horizon. Thus, the time of the second fire in the context of our management horizon $[0,Y]$ is $\tau_1+\tau_2$, the sum of the first sampled time of fire and the second sampled time of fire.

If $\tau_1+\tau_2>Y$, then the value of the forest over Y years is given by

$$\begin{aligned} J_Y = &\int_0^{\tau_1}\Big[B\big(A_1(t)\big)-h_1^*(t)\Big]e^{-rt}dt-\Big[D\Big(K\big(h_1^*(\tau_1),x_1^*(\tau_1)\big)\Big)+x_1^*(\tau_1)\Big]e^{-r\tau_1} \\ &+\int_0^{Y-\tau_1}\Big[B\big(A_2(t)\big)-h_2^*(t)\Big]e^{-rt}dt. \end{aligned} \tag{60}$$

Here, we take into account the cost associated with the first fire because it falls within $[0,Y]$. We do not take into account the costs associated with the second fire because because $\tau_1+\tau_2>Y$. Also notice that the limits of integration for the second integral are from 0 to $Y-\tau_1$. We begin at the time $t=0$ because the optimal control problem is solved over $[0,T]$. We only integrate up to $Y-\tau_1$ because there are only $Y-\tau_1$ years from the time of the first fire τ_1 to the end of the management horizon Y.

If $\tau_1+\tau_2\le Y$, then the value of the forest up to $\tau_1+\tau_2$ years is given by

$$\begin{aligned} &\int_0^{\tau_1}\Big[B\big(A_1(t)\big)-h_1^*(t)\Big]e^{-rt}dt-\Big[D\Big(K\big(h_1^*(\tau_1),x_1^*(\tau_1)\big)\Big)+x_1^*(\tau_1)\Big]e^{-r\tau_1} \\ +&\int_0^{\tau_2}\Big[B\big(A_2(t)\big)-h_2^*(t)\Big]e^{-rt}dt-\Big[D\Big(K\big(h_2^*(\tau_2),x_2^*(\tau_2)\big)\Big)+x_2^*(\tau_2)\Big]e^{-r\tau_2}. \end{aligned} \tag{61}$$

If $\tau_1+\tau_2=Y$, we are done since $\tau_2=Y-\tau_1$. However, if $\tau_1+\tau_2<Y$, once again, we must sample for another time of fire and solve our problem again. We

set our new initial condition for unburned acres,

$$A_3(0) = A_2(\tau_2) - K\big(h_2^*(\tau_2), x_2^*(\tau_2)\big), \tag{62}$$

solve our optimal control problem, and sample the next time of fire. We continue to do this until the sum of the sampled fire times is greater than or equal to Y.

Suppose that the n^{th} time of fire τ_n sampled gives $\tau_1 + \tau_2 + \cdots + \tau_n > Y$. Then, the value of the forest over Y years is given by

$$\begin{aligned} J_Y = & \int_0^{\tau_1} \Big[B\big(A_1(t)\big) - h_1^*(t)\Big] e^{-rt} dt - \Big[D\Big(K\big(h_1^*(\tau_1), x_1^*(\tau_1)\big)\Big) + x_1^*(\tau_1)\Big] e^{-r\tau_1} \\ & + \int_0^{\tau_2} \Big[B\big(A_2(t)\big) - h_2^*(t)\Big] e^{-rt} dt - \Big[D\Big(K\big(h_2^*(\tau_2), x_2^*(\tau_2)\big)\Big) + x_2^*(\tau_2)\Big] e^{-r\tau_2} \\ & + \cdots + \int_0^{Y-(\tau_1+\cdots+\tau_{n-1})} \Big[B\big(A_n(t)\big) - h_n^*(t)\Big] e^{-rt} dt, \end{aligned} \tag{63}$$

where $A_i(t)$ is governed by

$$A_i'(t) = \delta\big(\bar{A} - A_i(t)\big) \text{ with } A_i(0) = A_{i-1}(\tau_{i-1}) - K\big(h_{i-1}^*(\tau_{i-1}), x_{i-1}^*(\tau_{i-1})\big), \tag{64}$$

for $i = 2, \ldots, n$. Notice that the expenses from the final fire are not deducted from the value of the forest. This is because the final fire occurs outside of $[0, Y]$. If $\tau_1 + \tau_2 + \cdots + \tau_n = Y$ then

$$\begin{aligned} J_Y = \sum_{i=1}^{n} \Bigg[& \int_0^{\tau_i} \Big[B\big(A_i(t)\big) - h_i^*(t)\Big] e^{-rt} dt \\ & - \Big[D\Big(K\big(h_i^*(\tau_i), x_i^*(\tau_i)\big)\Big) + x_i^*(\tau_i)\Big] e^{-r\tau_i} \Bigg]. \end{aligned} \tag{65}$$

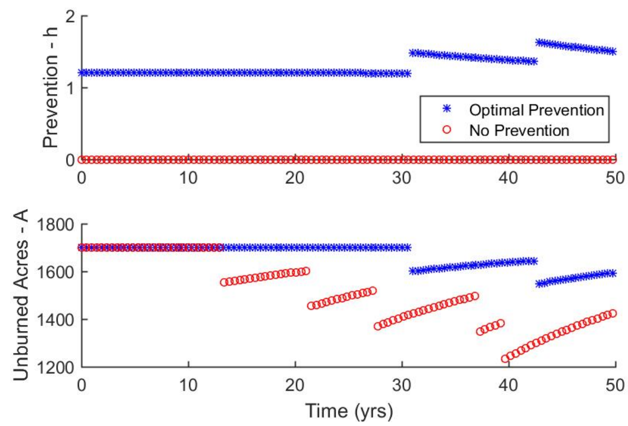

Figure 2: The top plot gives management prevention spending with optimal prevention and without prevention over a management horizon $Y = 50$ years The bottom plot gives the number of unburned acres. Every jump discontinuity represents a fire event.

Figure 2 provides one example of the management prevention schedule and number of unburned acres in one simulation where a sequence of fires is considered over $Y = 50$ years. Because we sample for the times of the fires, every fire sequence simulation will be different. The set of parameter values used are based on the values determined for the Las Conchas Fire and are found in Table 1. The only difference is that we choose a smaller value for the background hazard $b$and we let $b = 0.1$. These plots also show what the number of unburned acres might look like given no prevention management spending. This simulation is determined separately from the optimal case. This is because h determines y which is used to build the CDF used for sampling a time of fire. The jump discontinuities in the plots correspond to the different fire events. In the particular example in Figure 2 the no prevention management spending case 5 fires occur in 50 years and in the optimal prevention case 2 fires occur.

In order to create a more comprehensive picture concerning the effect of prevention management spending over a fixed management horizon for sequences

	Number of Fires		Value of Forest - \$ (M)	
Prevention	Optimal	None	Optimal	None
Mean	1.4	5.0	671	536
Median	1	5	677	556
Std.	1.1	2.4	34.0	111.7

Table 2: This table provides statistics concerning the average number of fires and average value of the forest over 50 years for 500 simulations.

of fires, we conduct many simulations and calculate statistics concerning the results. For our simulation study, a management horizon of $Y = 50$ years is considered and 500 trials are run. The set of parameter values, based on the values determined for the Las Conchas fire, used for the simulation study are found in Table 1, with the exception that now $b = 0.1$. Note in particular the large value for T and the initial condition $A_1(0)$. The initial condition $A_1(0) = \bar{A}$ only holds for the first solution of the optimal control problem in a single trial. Following that, the initial condition for the number of unburned acres following the first fire in a single trial are determined based on the sampled time of fire.

For the simulation study, 500 trials are conducted to determine the value of the forest J_Y over 50 years, given that an unknown number fires may occur in this time period for each trial. In addition to calculating value of the forest J_Y using the prevention management schedule found according to the optimal control problems, for comparison, we also calculate the value of the forest given that no money is spent on prevention management. It is important to note that these two cases are determined independently from one another. We also consider total prevention management spending and suppression spending in each case, in addition to the number of fires that occur in the management horizon. The results from the simulation study are discussed below.

Table 2 provides statistics concerning the distribution of the number of fires in the region encompassing the Santa Fe National Forest, Bandelier National

Monument, and the Valles Caldera Preserve in 50 years across 500 simulations. In the optimal prevention management case there are fewer large forest fires than in the no prevention case. In particular, the mean number of fires in the case with optimal prevention management is 1.4 and in the case with no prevention management spending is 5.0. Furthermore, the standard deviation of the number of fires is approximately 1 in the optimal prevention management case and greater than 2 in the no prevention management case. Moving from a case where there is no spending on prevention management to the case with optimal prevention management shown here, there is on average a 72% reduction in the number of fires that occur within 50 years. Hence, applying optimal prevention management spending reduces the risk of fire for a forest.

Additionally, Table 2 provide details concerning the distribution of the value of the forest J_Y over 50 years across 500 simulations. In the optimal prevention case the mean value of the forest over 50 years is \$671 million dollars and in the case without prevention management the mean value of the forest over 50 years is \$536 million dollars. Additionally, the standard deviation of the distribution for the value of the forest in the optimal prevention management case is 34.0, compared to 111.7 in the no prevention management case. That is, the standard deviation is three times larger in the case without prevention management spending compared to the case with optimal prevention management. Hence, the value of the forest over 50 years with multiple fires, is less variable and on average adds a value of \$1.2M per year to the value of the forest in the case of optimal prevention management compared to the case without prevention management.

We also calculate the total prevention management spending and suppression spending over 50 years. In the case without prevention management spending, on average \$236M is spent on forest fire suppression over 50 years. In the case with optimal prevention management spending, on average \$42M is spent on suppression over 50 years and \$65M is spent on prevention management spending over 50 years. That is, in the case applying the optimal prevention management spending, on average only \$107M is spent on prevention management and suppression combined. Thus, we can conclude that prevention management has the potential to offset high suppression costs and decrease spending overall.

Our results reveal that, on average, in the case of optimal prevention management spending there are fewer fires and an increased value of the forest in comparison to the case with no prevention management spending. Furthermore, the standard deviation around the average number of fires and value of the forest is much smaller in the optimal prevention management case in comparison to the no prevention management case. This suggests that using optimal prevention management spending is a less risky management option when compared to the case without prevention management spending. Additionally, we see that prevention management spending can offset high suppression costs and decrease the total amount of spending overall.

5 Conclusions

Rapid increases in wildfire suppression expenditures have prompted fire managers, scientists, and policy makers to investigate alternative approaches to managing wildfire. An increasingly popular alternative is fuels management which attempts to reduce wildfire risk and intensity through mechanical, chemical, biological or manual means, or by fire. This paper examines the economic tradeoffs between fuels management spending and suppression spending using a framework that recognizes how the inability to predict the timing of large fire events influences the riskiness of the two management options. We formulate an optimal prevention and suppression problem with stochastic time of fire and convert it to a deterministic optimal control problem using Reed's method. We present numerical results from our optimal control problem applied to a parameter set based on a recent fire event in New Mexico, a global parameter sensitivity analysis evaluating the impact of our parameters on the expected net value of the forest, and a simulation study concerning the effects of prevention management spending over a finite management horizon given an unknown sequences of fires.

We find that with the application of prevention management, the value of the forest is greater and less variable than in the case where prevention management spending is not applied to the forest. We also find that prevention spending lowers the number of devastating large fire events. The mean value of the forest over a 50 year time horizon in the no prevention management case is \$536M with a standard deviation of \$111.7M. In the case using prevention management determined by the successive application of our optimal control problem, we find the mean value of the forest over 50 years to be \$671M with a standard deviation of \$34.0M. This result illustrates that there are real economic costs associated with using funding for fuel management to fund immediate fire suppression.

Perhaps more surprisingly, we find that when optimal prevention management is employed, not only are high suppression costs drastically reduced, total spending on fire management (prevention and fire suppression) is less than the case without prevention management. In the case without prevention management spending, \$236M was spent on average on fire suppression over the course of 50 years. In the case with applying optimal prevention management spending, only \$42M was spent on average on suppression over 50 years and \$65M was spent on prevention management. By comparison, \$40M-\$50M was spent fighting the Las Conchas fire. In our work with unknown fire sequences, we observed an 88% reduction in suppression spending on average with prevention management, and a 55% reduction in spending overall. This result provides hope that a more careful integration of fire prevention into wildfire management plans may actually reduce the cost of these plans.

Our results clearly highlight the value of fuel management. This result arises even when we assume prevention expenditures only influence fire risk in the period they are incurred. If prevention spending has an impact on fire risk that extends beyond the current period, our results will under-value prevention and should be viewed as a lower bound. In spite of our results, expenditures on fire suppression will likely continue to outweigh hazardous fuel reduction expenditures. There are a number of factors not considered in our model that may explain this paradox. In our model, forest managers are forward-looking when they select suppression expenditures. However, forest managers acknowledge that public outcry during a large fire often prevents them from saving resources to fight future fires. To date, Congress has made up for any budget shortfalls that occur due to unexpected fire suppression expenditures. These factors suggest that forest mangers may choose fire suppression expenditures more myopically than our model suggests. These political economy considerations could be investigated by allowing the *ex ante* and *ex post* problems to be solved for different planning horizons. There are also additional liability considerations associated with prevention activities such as prescribed burning that our model does not consider. We leave these issues for future work.

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A Appendix: Global Parameter Sensitivity Analysis

The values chosen for our parameters are not all strictly data driven or from literature sources. Thus, we perform a global sensitivity analysis to determine which parameters have the most significant impact on the expected value of the forest, $J(h^*)$ [13, 16]. We use Latin Hypercube Sampling (LHS) and Partial Rank Correlation Coefficient (PRCC) analysis to determine the parameters to which the value of the objective functional evaluated at the optimal control h^* is most sensitive.

Table 3: The table below contains the lower and upper bounds of the parameter values to be used in our LHS/PRCC analysis. The baseline value for a given parameter is simply the average of the lower and upper bounds.

Parameter	Lower Bound	Upper Bound	Baseline Value
$\bar{A}$	1250	1900	1575
δ	0.025	0.075	0.05
B_1	0.01	0.05	0.03
r	0.03	0.05	0.04
b	0.1	0.2	0.15
c	0.01	0.75	0.38
v	0.5	1.5	1
k	5000	20000	12500
k_1	1	5	3
k_2	1	25	13

LHS was introduced in 1979 by M.D. McKay as an improved alternative to simple random sampling in Monte Carlo studies[17]. The LHS method provides similar accuracy as simple random sampling methods, but with fewer iterations, making it particularly useful for computationally expensive models [13, 16, 17].

There are 10 parameters in our optimal control problem that we will investigate. They are listed in the first column of Table 3. First, we must determine an appropriate range over which to investigate each of the parameters, or “inputs”. For each parameter we must choose an appropriate lower and upper bound for the parameter range; see Table 3. Next, we briefly discuss how these ranges were chosen. We note that while the values for the parameters chosen for the Las Conchas Fire example are included in these ranges, they do not serve as the baseline values for the parameters.

Our range for the size of forest parameter $\bar{A}$ is based on the sizes of national forests in the United States and is given in thousands of acres [43]. The forest regeneration rate δ parameter range is centered around our original choice of

$\delta = 0.05$ for Ponderosa Pine in the previous fire examples. Recall that for a particular fire event we determine a value for the flow of benefits parameter B_1 by solving the equation representing optimal fire suppression spending (43) for B_1 in terms of x^*. To determine a parameter range of B_1 to be used in the LHS/PRCC analysis, several fire events were considered and the range chosen is a reflection of the value of B_1 across these scenarios. The cost to built structures parameter c was chosen to capture the possibility of fire events in both isolated forest areas (smaller c) and well-developed forest areas (larger c). The range for the fire severity parameter k is chosen to capture a variety of high severity fires. The upper bound for the range is much larger than either of the choices in our examples because in our sensitivity analysis we are allowing for a range of values for k_1 and k_2, and not simply setting $k_1 = k_2 = 1$. The parameter range for k_1, associated with prevention management spending h, is chosen to be smaller than the range for parameter k_2 because, at a given point in time, less is spent on prevention management than on suppression. The choice for the background hazard parameter b was chosen to reflect the frequency with which large fire events may occur in a given area. The range for the discount rate parameter r is centered and varied around our original choice of $r = 0.04$. The range for the prevention management effectiveness parameter v, found in the hazard function, is centered and varied around our original choice of $v = 1$.

In order to properly use LHS, we must first verify that the output in question, $J(h^*)$, is monotonic with respect to each parameter[17]. That is, we solve our optimal control problem multiple times across the range of a given parameter, with all other parameters held at their baseline values, which is simply the average of the lower and upper bound for that parameter. We then verify that the value of the objective functional evaluated at the optimal control h^* is monotonic with respect to changes in the parameter. We repeated this process for every parameter and verified the monotonicity.

The LHS parameter matrix can now be generated. The LHS matrix is an $N \times 10$ matrix where N is the number of trials to run and 10 is the number of parameters to investigate. We assume a uniform distribution for all 10 parameters across their parameter ranges because the parameter ranges are not strictly data driven or from literature sources. Choosing $N = 50$, each parameter range is partitioned equally into 50 intervals and from each interval a sample is taken. Thus, each parameter is strategically sampled 50 times across its range and these 50 samples are stored in a column vector. The 10 column vectors, one for each parameter, make up the LHS matrix. For each individual parameter column vector in the LHS matrix, the sampled values are permuted so that they are not necessarily ordered. Thus, one row of the LHS matrix contains the parameter values to be used in a single trial of our optimal control problem.

Once the LHS matrix has been generated we solve our optimal control problem 50 times, once for each row vector of parameter values from the LHS matrix. For each trial, $J(h^*0$ is calculated. The mean of $J(h^*)$ for the 50 trials is $\mu = \$1,153$M. Given that the standard deviation for the output, $\sigma = \$525$M, is large in comparison to the mean, it is clear that the uncertainty present in the value of $J(h^*)$ is substantial. That is, variation in our choice of parameter values has a significant impact on $J(h^*)$. Hence, we follow this work with a PRCC sensitivity analysis to determine which parameters are the most significant contributors to this uncertainty.

Partial rank correlation coefficients (PRCCs) assess the degree of monotonicity between one input and the output, while controlling for the effects of the other inputs. That is, a PRCC is a sensitivity measure which allows us to assess nonlinear, but monotonic relationships between inputs and an output[13, 16]. A PRCC is calculated for each parameter investigated.

Table 4: In this table, the partial rank correlation coefficients for each parameter associated with the output $J(h^*)$, along with the corresponding p-values, are listed. Using a significance level of $\alpha = 0.05$ we see that 5 of the 10 parameters investigated are significantly different from zero. They are highlighted in yellow.

Parameter	PRCC	p-value
$\bar{A}$	0.88	$<< 0.05$
δ	-0.04	0.82
B_1	0.99	$<< 0.05$
r	-0.86	$<< 0.05$
b	-0.09	0.57
c	-0.37	0.02
v	0.08	0.61
k	-0.30	0.052
k_1	-0.10	0.54
k_2	0.37	0.02

The PRCC for each parameter and associated p-values in Table 4 are calculated using the MATLAB function `partialcorr()`. The p-values are used to assess whether or not the PRCCs are significantly different from zero. Using a significance level of $\alpha = 0.05$, we see that 5 of our 10 parameters have PRCCs significantly different from zero. These parameters include the size of the forest parameter $\bar{A}$, the flow of benefits parameter B_1, the discount rate parameter r, the nontimber damage cost parameter c, and the suppression spending effectiveness parameter k_2. We interpret this to mean that the parameters which have PRCCs significantly different from zero have a significant impact on $J(h^*)$.

Now that we know which parameters have a significant impact on the output, we would like to make a comparison of these significant parameters to see which ones have the strongest impact, in magnitude, on $J(h^*)$. To determine if a given parameter has a greater impact on the output than another, we must determine if there are significant statistical differences in their corresponding PRCCs. In

Table 5: This table lists the PRCCs and their corresponding Fisher transforms for the parameters which were shown to have the most impact on the value of the objective functional evaluated at the optimal control h^*.

Parameter	PRCC γ	Fisher Transform γ'
$\bar{A}$	0.88	1.38
B	0.99	2.53
r	-0.86	-1.31
c	-0.37	-0.39
k_2	0.37	0.39

order to perform statistical comparison tests for PRCCs, we must first apply the following log transformation to each PRCC:

$$\gamma' = \frac{1}{2} \ln \left| \frac{1+\gamma}{1-\gamma} \right|, \tag{66}$$

where γ is the original PRCC and γ' is the transformed PRCC[16, 3]. The log transformed PRCC γ' is known as the Fisher tranform and is approximately Gaussian $\mathcal{N}(\mu, \sigma^2)$ with

$$\mu = \frac{1}{2} \ln \left| \frac{1+\gamma}{1-\gamma} \right| \text{ and } \sigma^2 = \frac{1}{N-3-p}, \tag{67}$$

where N is the number of trials and p is the number of parameters controlled for when the PRCC is calculated. Table 5 gives the Fisher transformed PRCCs for the parameters whose PRCCs are significantly different from zero.

We can compare the values of two PRCCs by examining the z-statistic

$$z = \frac{\gamma_1' - \gamma_2'}{\sqrt{\frac{1}{N_1-3-p_1} + \frac{1}{2-3-p_2}}}, \tag{68}$$

which follows a $\mathcal{N}(0,1)$ distribution. Here, $N_1 = N_2 = 50$ is the number of trials and the value p_i, $i = 1, 2$, represents the number of parameters controlled for when the PRCC γ_i is calculated [16]. For our problem, $p_1 = p_2 = 9$ since we are investigating 10 parameters. We are most interested in determining which parameters have the largest impact on the output in magnitude, regardless of whether that impact is positive or negative. This guides the development of the family of hypotheses we wish test to determine the ranking of the significant parameters.

To properly rank the PRCCs according to their impact on the output $J(h^*)$ in magnitude, we must perform multiple pairwise comparison tests. In particular, we test the null hypothesis that all PRCCs are equal

$$H_0 : |\gamma'_{\bar{A}}| = |\gamma'_{B_1}| = |\gamma'_c| = |\gamma'_{k_2}| = |\gamma'_r| \tag{69}$$

against the alternative hypotheses

$$H_A : |\gamma'_i| \neq |\gamma'_j|, \tag{70}$$

for every pair $(i, j) \in \{\bar{A}, B_1, c, k_2, r\}$ where $i \neq j$. Thus, we have a family of $\binom{5}{2} = 10$ pairwise hypothesis tests to perform in order to effectively rank our 5 significant parameters.

When performing multiple comparison tests we must be careful to consider the increased likelihood of a rare event; that is, when considering multiple tests, we are more likely to reject the null hypothesis when it is true, a type I error. Given that we are performing 10 hypothesis tests and have chosen a significance level of $\alpha = 0.05$, the probability that we reject at least one of the null hypotheses (i.e. the probability of at least one rare event) is

$$
\begin{aligned}
P(\geq 1 \text{ significant event}) &= 1 - P(0 \text{ significant events}) \\
&= 1 - (1 - 0.05)^{10} \\
&= 0.40126. \qquad (71)
\end{aligned}
$$

In other words, using a significance level of $\alpha = 0.05$ for each of the 10 tests, the probability of at least one significant event (at least one rejection of the null hypothesis) is approximately 40%. This is known as the familywise error rate (FWER). We would like to control the FWER on our family of hypothesis tests in order to control the number of false positives. To do so, we need to differentiate between a per test significance level $\alpha[PT]$, read "alpha per test," and a per family significance level $\alpha[PF]$, read "alpha per family." Given a family of hypothesis tests we would like to control the familywise error rate at the level of $\alpha[PF] = 0.05$. The FWER for a given $\alpha[PT]$ is given by

$$
\alpha[PF] = 1 - (1 - \alpha[PT])^{C}, \qquad (72)
$$

where C is the number of hypothesis tests [1]. This is known as the Sidak equation which can be rewritten to give the $\alpha[PT]$ for a given $\alpha[PF]$:

$$
\alpha[PT] = 1 - (1 - \alpha[PF])^{\frac{1}{C}}. \qquad (73)
$$

Thus, given that we want $\alpha[PF] = 0.05$ and we are performing $C = 10$ tests, we can solve for $\alpha[PT]$. However, to determine $\alpha[PT]$ we use the simpler Bonferroni approximation:

$$
\alpha[PT] \approx \frac{\alpha[PF]}{C}. \qquad (74)
$$

Table 6: The table below contains the results of the hypothesis tests to determine the ranking of our significant parameters according to their impact on the output. To control the FWER, using the Bonferroni approximation, we use a per test significance level of 0.005 to determine whether or not to reject the null hypothesis.

Hypothesis Test Results - $J(h^*)$		
Alternative Hypothesis	z-statistic	Conclusion
$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_{B_1}\rvert$	5.540	reject null
$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_{c}\rvert$	4.304	reject null
$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_{k_2}\rvert$	4.304	reject null
$\lvert\gamma'_{\bar{A}}\rvert \neq \lvert\gamma'_{r}\rvert$	0.359	FAIL TO REJECT
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_{c}\rvert$	9.843	reject null
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_{k_2}\rvert$	9.843	reject null
$\lvert\gamma'_{B_1}\rvert \neq \lvert\gamma'_{r}\rvert$	5.899	reject null
$\lvert\gamma'_{c}\rvert \neq \lvert\gamma'_{k_2}\rvert$	0	FAIL TO REJECT
$\lvert\gamma'_{c}\rvert \neq \lvert\gamma'_{r}\rvert$	3.944	reject null
$\lvert\gamma'_{k_2}\rvert \neq \lvert\gamma'_{r}\rvert$	3.944	reject null

The Bonferroni approximation is the linear approximation of the Sidak equation and its use is well-established in the literature as a procedure to control FWER [1].

Thus, to control the type I error for the family of 10 hypotheses we are testing at a significance level of $\alpha[PF] = 0.05$, we use a per test significance level of

$$\alpha[PT] \approx \frac{\alpha[PF]}{C} = \frac{0.05}{10} = 0.005. \tag{75}$$

Next, we perform our family of hypothesis tests, with $\alpha[PT] = 0.005$. We are testing the null hypothesis (69) against the alternative hypotheses that a pair of PRCCs are not equal. There are 10 pairs of PRCCs as we are ranking 5 parameters. The z-statistic to test the hypotheses is given by equation (68).

The results of the 10 pairwise comparison tests are included in Table 6. The z-score associated with $\alpha[PT] = 0.005$ is 2.807. Therefore, if the z-statistic for a given hypothesis test is is greater than 2.807, then the null hypothesis is rejected and if the z-statistic is less than 2.807, then we fail to reject the null hypothesis. To reject the null hypothesis means that the two PRCCs considered in the alternative hypothesis are significantly different from one another. Thus, the parameter with the larger PRCC in absolute value has a greater impact on the output $J(h^*)$. To fail to reject the null hypothesis means that there is not enough evidence to conclude that the two PRCCs being compared are significantly different and hence no conclusions about which parameter has a greater impact on the output can be drawn. The full results of our hypothesis tests are summarized in Table 6, and we now explain specific comparisons in detail.

First, we test the null hypothesis (69) against the alternative hypothesis $|\gamma'_{\bar{A}}| \neq |\gamma'_{B_1}|$. With a z-statistic of $5.540 > 2.807$, we reject the null hypothesis and conclude that the parameter B_1 has a greater impact in magnitude on the output than the parameter $\bar{A}$. Here, we note that the parameter B_1 has the PRCC closest to 1 in magnitude and $\bar{A}$ has the second largest PRCC. Thus, it is not surprising, and is expected, that when considering the alternative hypotheses $|\gamma'_{B_1}| \neq |\gamma'_c|$, $|\gamma'_{B_1}| \neq |\gamma'_{k_2}|$, and $|\gamma'_{B_1}| \neq |\gamma'_r|$, we also reject the null hypothesis. Hence, we conclude that the impact of B_1 on the value of the output is also greater than the impact of the parameters c, k_2 and r on the output. Therefore, B_1, the flow of benefits parameter, is the parameter which has the greatest impact on the output in magnitude.

Next, we compare the PRCCs of the parameters $\bar{A}$ and r. With a z-statistic of $0.359 < 2.807$ we fail to reject the null hypothesis. That is, we cannot make any conclusions about whether $\bar{A}$ or r has a greater impact on $J(h^*)$ in magnitude. Thus, the parameters $\bar{A}$ and r, which were shown to have significantly less impact on the output than the parameter B_1, were not shown to be significantly different from one another when considering their impact on the output in absolute value. The z-statistics for comparing the impact of $\bar{A}$ against c and $\bar{A}$ against k_2 are equal because, in terms of absolute value, c and k_2 have the same PRCC. Hence, with a z-statistic of $4.304 > 2.807$ we reject the null hypothesis and conclude that the impact of $\bar{A}$ on the output is greater than the impact of c and greater than the impact of k_1 on the output.

Once again, as the PRCCs for c and k_2 are equal in absolute value, comparing the impact of r against c on the output is equivalent to comparing the impact of r against k_2 on the output. Similarly we can conclude that in terms of magnitude, the parameters c and k_2 have a similar impact on the output $J(h^*)$. In summary, we use the Bonferroni approximation to determine a per test significance level for the family of hypothesis tests used in order to rank the parameters according to their impact on the value of the objective functional. We conclude that the parameter B_1 has the strongest impact on the expected net present value of the forest, followed by parameters $\bar{A}$ and r, followed by parameters c and k_2.