QUANTUM COMPUTATION OF SCATTERING AMPLITUDES

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QC

Nature is **quantum**.

- hard to understand for us classical beings.
- no complete description of what **quantum** means.
  - need (classical) concept of *measurement* to make sense of **quantum**.

By understanding **quantum**, we hope to be able to take advantage of *everything* Nature has to offer.

- ▶ One such advantage seems to be in IT.

Information and entropy

What does information have to do with Physics?

- Computers process information. They need power to operate. They get *hot.* Why?
- Thermodynamics: processes in Nature are *irreversible*. Entropy always increases, heat flows from hot to cold, etc.
- But laws of Nature are invariant under time reversal.
  - At a fundamental level, all processes are *reversible* ($\Delta S = 0$).

Is processing of information an irreversible (dissipative) process?

ANSWER: No!

EXAMPLE: The NAND gate, $(a,b) \mapsto \overline{(a \wedge b)}$

TRUTH TABLE:

a	b	$a \wedge b$	$\overline{(a \wedge b)}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

▶ Irreversible!

Add a bit and do *Toffoli gate*, instead..

$$T : (a, b, c) \mapsto (a, b, c \oplus (a \wedge b))$$

▶ flips $a \wedge b$, if $c = 1$, $\therefore$ NAND gate.

TRUTH TABLE:

a	b	c	$a \wedge b$	$c \oplus (a \wedge b)$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	0

▶ Reversible! ($T^2 = \mathbb{I}$, so $T^{-1} = T$)

Then what costs energy (entropy)?

ANSWER: The *erasure* of information.

Landauer's principle (1961)

Erasing 1 bit of information requires entropy $\Delta S = k_B \log 2$.

- Minimum cost in operating a computer.
- Present computers far from this limit ($\Delta S \sim 500 k_B \log 2$).
- As computers get smaller, this limit will become significant.

Do we have to erase information?

ANSWER: No!

- ▶ At the end of a reversible computation, computer can reverse all steps and return to its initial state.
  - No junk to dispose of!
  - No energy loss!
  - No entropy generated!

With computers near Landauer limit, all information processing will have to be done *reversibly*, otherwise wires will melt.

Quantum information

You can extract information from a classical system without disturbing it.

- Can't do that with a quantum system.

- FOOD FOR THOUGHT: Aren't all systems quantum?

You measure observable A and system collapses to an eigenstate of A.

- Worse: if $[A, B] \neq 0$, then measurement of A will influence subsequent measurement of B.

Outcome of measurement is random:

- Let $A|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$.
  If system is in state $|\Psi\rangle$, then outcome of measurement of A is λ_i with probability

$$P_i = |\langle\lambda_i|\Psi\rangle|^2$$

  - State $|\Psi\rangle$ collapses to $|\lambda_i\rangle$.
  - We have no way of figuring out $|\Psi\rangle$.

Qubits

Classical information comes in *bits*, which take values 0 or 1.

▶ Quantum information comes in *qubits*, i.e.,

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

Probabilistic outcome: A measurement will project onto $|0\rangle$ or $|1\rangle$ with probability, respectively,

$$P_0 = |a|^2 \ , \quad P_1 = |b|^2$$

EXAMPLES: spin-1/2 particle ($|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$).
photon polarization ($|0\rangle = |L\rangle$, $|1\rangle = |R\rangle$).

With N qubits, state is superposition of $\{|x\rangle, x = 0, 1, \dots, 2^N - 1\}$

- x in binary notation, e.g., for $N = 3$,

$$x = 000, 001, 010, 011, 100, 101, 110, 111$$

Quantum computation

1. Prepare initial state of N qubits.
2. Evolve state by applying a string of *quantum gates*
   - evolution operators U ($2^N \times 2^N$ unitary matrices, $U^\dagger U = \mathbb{I}$)
3. perform a measurement on the final state.

Efficiency: with $N = 100$ qubits, we can naturally (through quantum evolution) implement $10^{30} \times 10^{30}$ matrices ($2^{100} \sim 10^{30}$).

– Try that on a classical computer!

– Hilbert spaces are enormous!

Errors

A quantum algorithm is useful and goes beyond classical, if it can take advantage of *nonlocal correlations* (entanglement).

- $|01\rangle$ and $|10\rangle$ are not entangled.
- $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is entangled.
  - Einstein-Podolsky-Rosen (EPR) paradox.
- Extremely fragile correlations.
  - decay *very* rapidly, due to interactions with environment.
  - some (most) information lies in correlations with environment and cannot be accessed.

Schrödinger's cat

$$|\text{cat}\rangle = \frac{1}{\sqrt{2}}(|\text{dead}\rangle + |\text{alive}\rangle)$$

Didn't like it: all cats he had observed were either $|\text{dead}\rangle$ or $|\text{alive}\rangle$.

Why?

Cat interacts with environment and information is *immediately* transfered to correlations with environment.

- Environment *measures* cat continuously, projecting it onto states we are familiar with (decoherence).

For QC, we need a $|\text{cat}\rangle$-like state, except not as large.

- Need to deal with errors due to decoherence.
  - the state will generically interact with environment and decay very rapidly
  - computer will immediately crash.

Shor's algorithm (1994)

A milestone.

- Showed that QC is by far superior to a classical computer, because it can find prime factors of a given number very efficiently.

Intractable problem: hard to find solution, but easy to verify once found.

- Let $N = pq$, $p \sim 2^m$, $q \sim 2^n$.
- Need $\sim mn \sim \log p \log q \sim (\log N)^2$ steps (time) to verify $pq = N$.
- Given N, to find p and q, best algorithm (*number field sieve*) takes *superpolynomial* time

$$t \approx C \exp\left\{ \left[\frac{64}{9} \log N (\log \log N)^2 \right]^{1/3} \right\}$$

Experimentally, for $N \sim 10^{130}$, using a few hundred workstations, $t \approx 1$ month, so $C \approx 1.5 \times 10^{-18}$ months.
For a 400-digit number, we need

$$t \approx 2.6 \times 10^{11} \text{ months} \sim 10^{10} \text{ years} \sim \text{age of the Universe!}$$

Shor's quantum algorithm takes *polynomial* time to run,

$$t \approx C'(\log N)^3$$

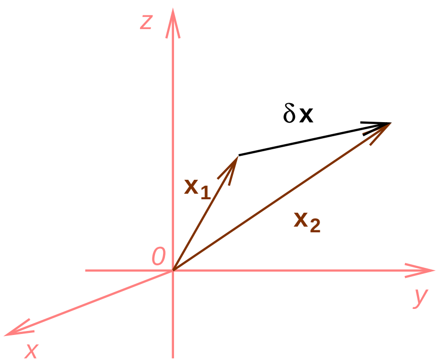
If a (future!) QC matches the performance of a classical computer for a 130-digit number, then $C' \approx 4 \times 10^{-8}$ months, and for a 400-digit number, we need

$$t \approx 29 \text{ months} \sim 2.5 \text{ years}$$

Huge (exponential) improvement!

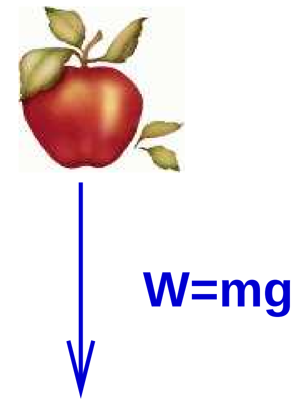
QFT

NEWTONIAN SPACE AND TIME



$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2$$

invariant under rotations

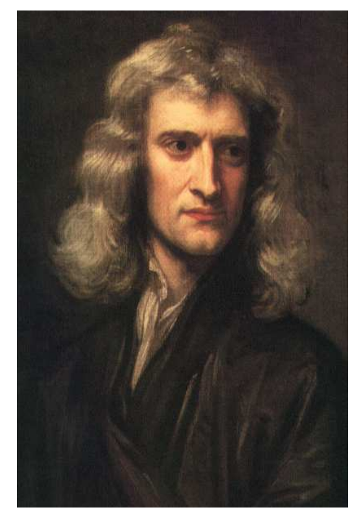


inertial observers feel no forces.

$$\vec{F} = m\vec{a}$$

invariant under

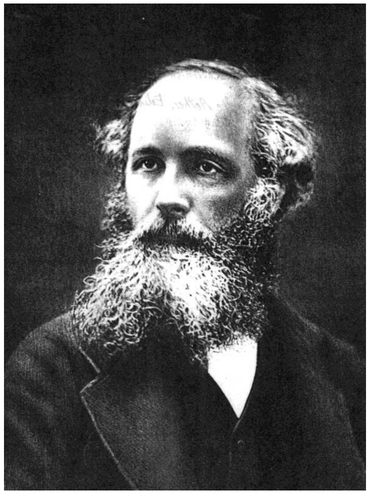
$$\vec{x} \to \vec{x} - \vec{v}t$$



(transformation law between inertial observers - Galilean).

★ Newton is inertial; apple isn't ($a = -g$).

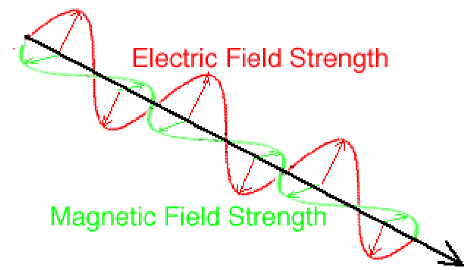
ELECTROMAGNETISM



Unification of electricity and magnetism (Maxwell)

$\Rightarrow$ waves

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0$$



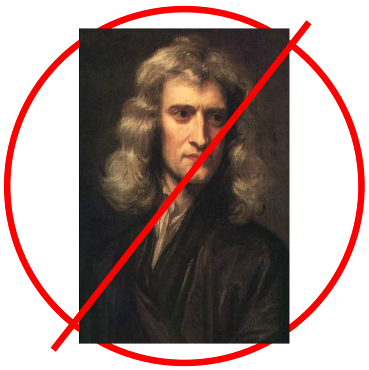
they travel at speed of light

$$c = 3 \times 10^8 m/s$$

regardless of frame of reference

$\therefore$ incompatible with Newton's laws.

Lorentz transformation (boost in x-direction):

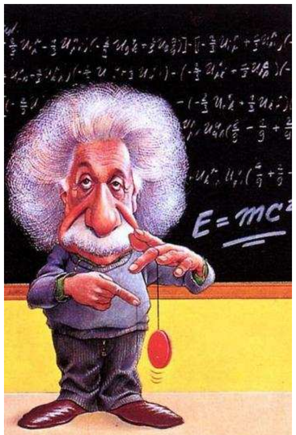


$$x \to \gamma(x - vt) \;\;,\;\; t \to \gamma(t - vx/c^2)$$

($\gamma < 1$ - contraction) known before Einstein.

★ Radiation consists of *fields.*

SPACETIME

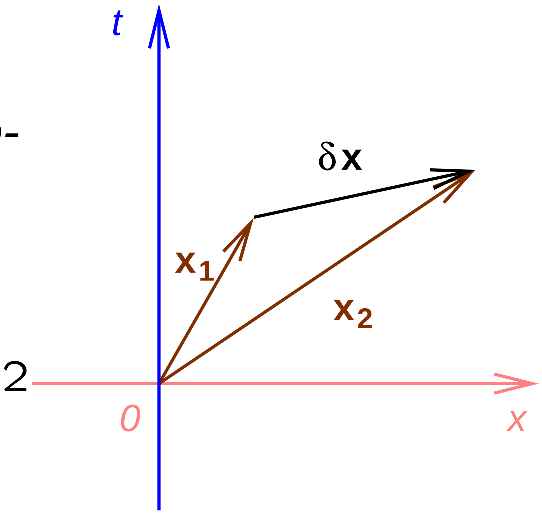


Einstein: space and time mix, form *space-time*

Invariant distance:

$$\delta s^2 = -c^2\delta t^2 + \delta x^2 + \delta y^2 + \delta z^2 = -c^2\delta\tau^2$$

τ: proper time.



SPECIAL RELATIVITY

Far-reaching consequences:

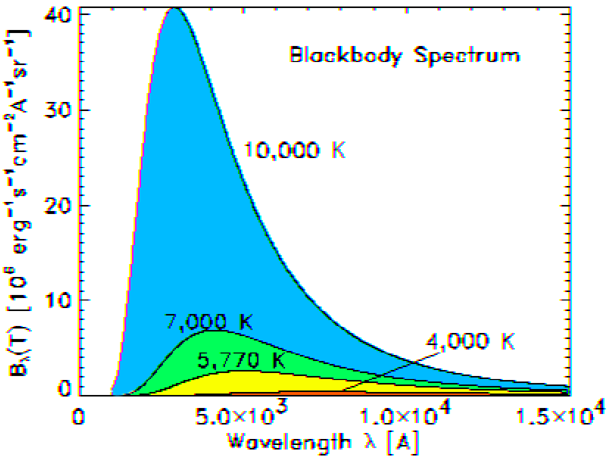
$$E = mc^2$$

QUANTUM MECHANICS

Blackbody spectrum:

$$I(\nu) = \frac{8\pi k T \nu^2}{c^3}$$

$\longleftarrow$ WRONG!



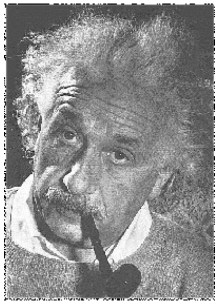
Planck in an act of despair proposes (light emitted as quanta)

$$E = h\nu$$

changes an integral to a sum:

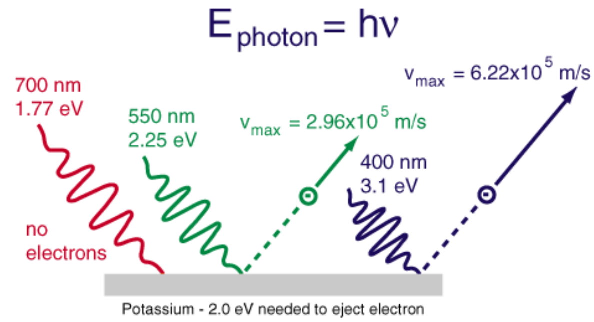
$$I(\nu) = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1}$$

$\longleftarrow$ CORRECT!

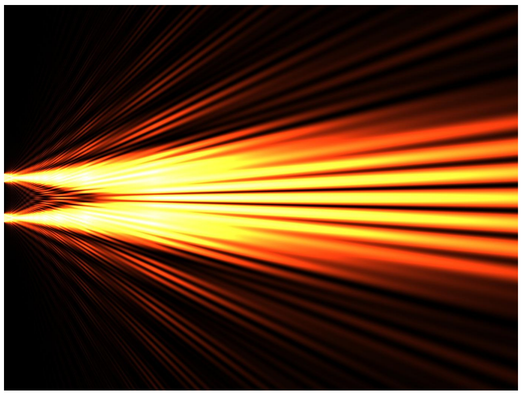


Einstein explains photoelectric effect (1921 Nobel Prize)

- light is absorbed as quanta.



Photoelectric effect



QUESTION: does light travel as particles (photons)?

- hard to swallow: photons can't be *hard balls* - they interfere!

Einstein realized the consequences (unpredictability) *before* Heisenberg's Uncertainty Principle

$$\Delta p \Delta x \geq \hbar$$

- didn't like it


"God does not play dice with the Universe"

QUANTUM MECHANICS + RELATIVITY

Klein-Gordon equation (discovered by Schrödinger)

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = m^2\Psi$$

- doesn't work (negative probabilities)

Dirac equation

$$i\gamma^\mu\partial_\mu\Psi = m\Psi$$

- leads to hole theory nonsense.

Finally, Quantum Field Theory - a triumph!

Ψ *is not a wavefunction (probability amplitude), but a* ***field****,*
similar to electric and magnetic fields.

- ◇ Explains Pauli exclusion principle
- ◇ Predicts anti-matter

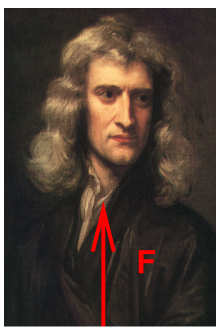
GRAVITY + RELATIVITY

no forces on a satellite

Einstein: The apple is the inertial observer, <u>not</u> Newton!

- ▶ Newton feels a force, apple doesn't.

Principle of Equivalence



Apple travels along *geodesic*

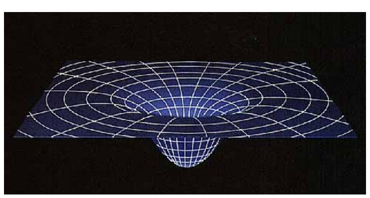
$$x \approx -\frac{1}{2}gt^2$$

plane going from London to NYC.

$$ds^2 = -c^2 d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu$$

mass (energy) *creates* spacetime

- ▶ time "warp" factor:

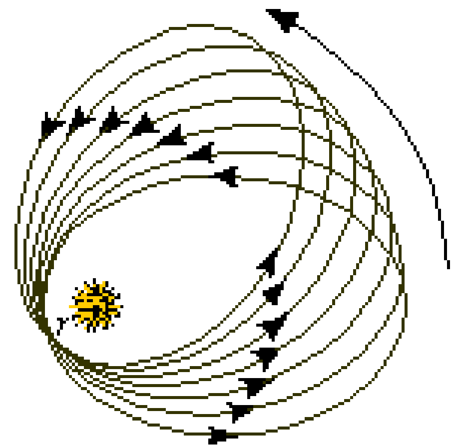


$$g_{00} \simeq 1 - \frac{2gR}{c^2} = 1 - 1.4 \times 10^{-9} = 0.9999999986$$

GENERAL RELATIVITY

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Instant success:



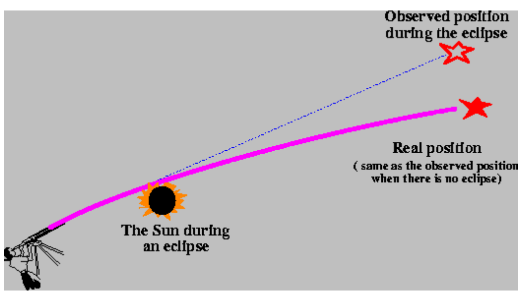
★ Mercury precession

observed: $\Delta\phi = 5,601''$/century.

geocentric system effects: $5,026''$/century.

other planets: $532''$/century.

Einstein: $43''$/century.



★ bending of light by Sun

$$\Delta\theta \simeq \frac{4GM_\odot}{c^2 R_\odot} \simeq 1.75''$$

observed by Eddington (1919)

★ Big-bang nucleosynthesis (BBN) (Universe ∼1 sec old)

QUANTUM MECHANICS + ELECTROMAGNETISM

coupling (fine-structure) constant

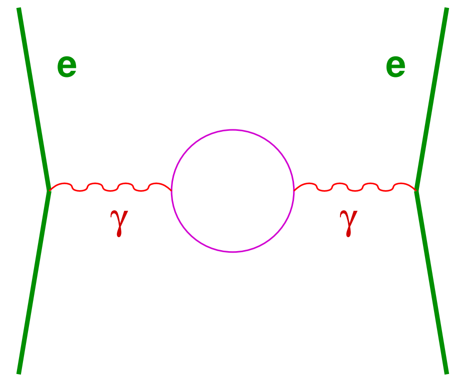
$$\alpha = \frac{e^2}{2hc} = \frac{1}{137}$$

QED - infinities

vacuum polarization modifies Coulomb Law

$$V(r) = \frac{e(r)}{4\pi r} \ , \ \ e(r) = e\left\{1 + \frac{2\alpha}{3\pi}\ln\frac{\lambda_e}{r} + \dots\right\}$$

for $r \ll \lambda_e = h/m_e c$ (Compton wavelength).

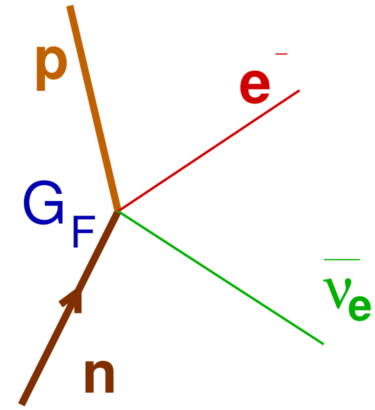


★ running coupling constant (vacuum is a dielectric)

OTHER FORCES

WEAK: β decay

$$n \to p + e^- + \bar{\nu}_e$$



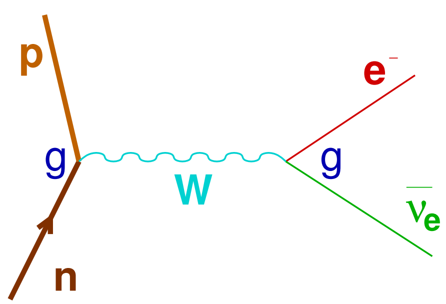
point interaction with coupling (Fermi)

$$G_F = 1.166 \times 10^{-5} (\hbar c)^3 / GeV^2$$

needs modification at high energies (infinite probabilities)

Introduce weak (W) boson (massive) mediating interaction.

$$\frac{G_F}{(\hbar c)^3} = \frac{\sqrt{2}\, g^2}{8 m_W^2 c^4} \ , \quad m_W = 80.425 \ GeV/c^2$$

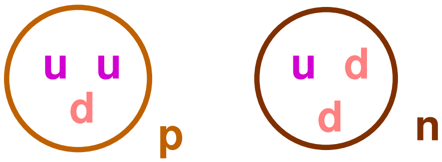


weak "fine-structure" constant (dimensionless):

$$\alpha_W = \frac{g^2 \hbar^3}{4\pi c} = \frac{1}{29}$$

$\alpha / \alpha_W = \sin^2 \theta_W$ (Weinberg angle).

STRONG: Quantum Chromodynamics (QCD)



nucleons consist of quarks.

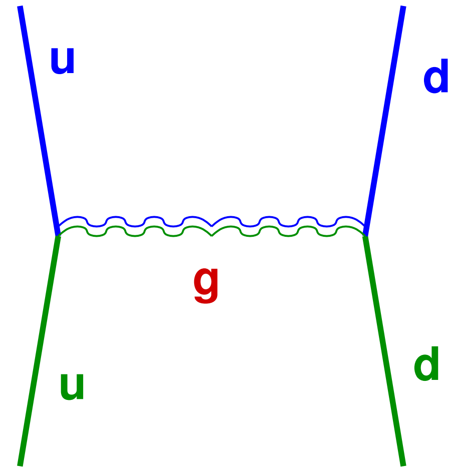
Quarks and gluons have colors

▶ cannot be free (directly detected)

Only color neutral objects can be "seen" (nucleons, mesons, etc)

Scale:

$$\Lambda_{QCD} = 0.236\ GeV$$



interaction via gluons.

★ asymptotic freedom

At high energies: coupling constant *decreases*

▶ spin-1 *charged* bosons (gluons) make vacuum paramagnetic

[Politzer; Gross and Wilczek - Nobel Prize 2004]

E&M + WEAK + STRONG = **STANDARD MODEL**

QUANTUM MECHANICS + GRAVITY

Newton's constant G is like weak (Fermi) constant G_F.
needs to be expressed in terms of a massive boson

$$G \sim \frac{g^2}{m_P^2} \;\; , \;\; m_P = \sqrt{\frac{\hbar c}{G}} \simeq 10^{19} \, GeV/c^2$$

P for Planck.
m_P is the scale where quantum effects are expected to become important.

QC OF QFT

Scattering

Scattering process

$$|\mathbf{1}, \vec{k}_1; \cdots ; \mathbf{M}, \vec{k}_M\rangle_{t\to-\infty} \to |\mathbf{1}', \vec{k}'_1; \cdots ; \mathbf{N}', \vec{k}'_N\rangle_{t\to+\infty}$$

Scattering amplitude

$$\mathcal{A} = {}_{t\to+\infty}\langle \mathbf{M}, \vec{k}_M; \cdots ; \mathbf{1}, \vec{k}_1 | \mathbf{1}', \vec{k}'_1; \cdots ; \mathbf{N}', \vec{k}'_N\rangle_{t\to+\infty}$$

To calculate cross section $|\mathcal{A}|^2$, apply quantum algorithm:

[S. P. Jordan, et al., Science **336***, 1130 (2012).]*

1. Prepare initial state $|\mathbf{1}, \vec{k}_1; \cdots ; \mathbf{M}, \vec{k}_M\rangle_{t\to-\infty}$ (eigenstate of *free* theory).
2. *Adiabatically* evolve it to an interacting theory eigenstate.
3. Evolve it for the duration of scattering with (unitary) quantum gates.
4. *Adiabatically* evolve it back to free theory state.
5. Measure the number of momentum modes (detector simulation).

Exponential improvement over lattice field theory on classical computers!

Harmonic oscillator

Set $\hbar = c = 1$.

Operators q (position) and p (momentum) satisfy commutation relations

$$[q, p] = i$$

The Hamiltonian is (setting the mass $m = 1$, for simplicity)

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2$$

where I added a constant to shift the ground state energy to zero.

Next, introduce

$$a \equiv \sqrt{\frac{\omega}{2}}q + \frac{i}{\sqrt{2\omega}}p$$

We have

$$[a, a^\dagger] = 1$$

$$q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \quad , \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

Therefore, with *normal ordering*

$$H = \omega a^\dagger a$$

Eigenvalue problem solved

$$H|n\rangle = E_n|n\rangle \quad , \qquad E_n = n\omega$$

Can be simulated with N qubits, if mapped onto basis states $|n\rangle$, $n = 0, 1, \ldots, 2^N - 1$.

<u>Alternative simulation</u>

Working in the q representation, we have

$$p = -i\frac{d}{dq} \quad , \quad a = \sqrt{\frac{\omega}{2}}q + \frac{1}{\sqrt{2\omega}}\frac{d}{dq}$$

The ground state obeys

$$a\Psi_0(q) = \sqrt{\frac{\omega}{2}}q\Psi_0(q) + \frac{1}{\sqrt{2\omega}}\Psi_0'(q) = 0$$

$\therefore \quad \Psi_0(q) = Ce^{-\omega q^2/2}$.

This Gaussian can be simulated by N qubits.

Other states can then be generated.

For 1st excited state, introduce ancillary qubit and define

$$H_1 = a^\dagger|1\rangle\langle 0| + a|0\rangle\langle 1|$$

We have

$$H_1\Psi_0|0\rangle = \Psi_1|1\rangle \quad , \quad H_1\Psi_1|1\rangle = \Psi_0|0\rangle$$

therefore

$$e^{-iH_1\pi/2}\Psi_0|0\rangle = -i\Psi_1|1\rangle$$

Free scalar field theory

Now consider a free scalar field $\phi(\vec{x})$, where $\vec{x} \in \Omega$, which is a cubic lattice in d spatial dimensions with length L and lattice spacing a

$$\Omega = a\mathbb{Z}^d_{L/a}$$

Let $\vec{a}_i$ $(i = 1, \ldots, d)$ denote the basis of the lattice Ω $(|\vec{a}_i| = a)$.

- To compare with Nature, presumably we need to take the *continuum limit*

$$a \to 0$$

Let $\pi(\vec{x})$ be the conjugate field, satisfying commutation relations

$$[\phi(\vec{x}), \pi(\vec{y})] = \frac{i}{a^d}\delta_{\vec{x},\vec{y}}$$

The discretized gradient is defined by

$$\nabla_i \phi(\vec{x}) = \frac{\phi(\vec{x} + \vec{a}_i) - \phi(\vec{x})}{a} \quad (i = 1, \ldots, d)$$

The discretized Laplacian is

$$\nabla^2\phi(\vec{x}) = \sum_{i=1}^{d} \nabla_i^2\phi(\vec{x}) = \sum_{i=1}^{d} \frac{\phi(\vec{x}+\vec{a}_i) + \phi(\vec{x}-\vec{a}_i) - 2\phi(\vec{x})}{a^2}$$

The Hamiltonian is

$$H_0 = \frac{a^d}{2} \sum_{\vec{x}\in\Omega} \left(\pi^2(\vec{x}) + \phi(\vec{x})(-\nabla^2 + m^2)\phi(\vec{x})\right)$$

Next, introduce the dual lattice $\Gamma = \frac{2\pi}{L}\mathbb{Z}^d_{L/a}$

and the annihilation operator

$$a(\vec{k}) = a^d \sum_{\vec{x}\in\Omega} e^{-i\vec{k}\cdot\vec{x}} \left[\sqrt{\frac{\omega(\vec{k})}{2}}\phi(\vec{x}) + \frac{i}{\sqrt{2\omega(\vec{k})}}\pi(\vec{x})\right]$$

where $\vec{k} \in \Gamma$, and

$$\begin{aligned} \omega^2(\vec{k}) &= e^{-i\vec{k}\cdot\vec{x}}(-\nabla^2 + m^2)e^{i\vec{k}\cdot\vec{x}} \\ &= m^2 + \frac{4}{a^2}\sum_{i=1}^{d} \sin^2\frac{k_i a}{2} \end{aligned}$$

We have

$$[a(\vec{k}), a^\dagger(\vec{k}')] = L^d \delta_{\vec{k},\vec{k}'}$$

$$\begin{aligned} \phi(\vec{x}) &= \frac{1}{L^d} \sum_{\vec{k}\in\Gamma} e^{i\vec{k}\cdot\vec{x}} \frac{1}{\sqrt{2\omega(\vec{k})}} [a(\vec{k}) + a^\dagger(-\vec{k})] \\ \pi(\vec{x}) &= -\frac{i}{L^d} \sum_{\vec{k}\in\Gamma} e^{i\vec{k}\cdot\vec{x}} \sqrt{\frac{\omega(\vec{k})}{2}} [a(\vec{k}) - a^\dagger(-\vec{k})] \end{aligned}$$

Therefore,

$$H_0 = \sum_{\vec{k}\in\Gamma} \mathcal{H}(\vec{k}) \ , \quad \mathcal{H}(\vec{k}) = \frac{1}{L^d} \omega(\vec{k}) a^\dagger(\vec{k}) a(\vec{k})$$

- All terms commute with each other.
- Each represents a harmonic oscillator.

Eigenvalue problem solved

$$H|\{n(\vec{k}),\vec{k}\in\Gamma\}\rangle = E_{\{n(\vec{k}),\vec{k}\in\Gamma\}}|\{n(\vec{k}),\vec{k}\in\Gamma\}\rangle$$

where

$$E_{\{n(\vec{k}),\vec{k}\in\Gamma\}} = \sum_{\vec{k}\in\Gamma} n(\vec{k})\omega(\vec{k})$$

Physical interpretation: $n(\vec{k})$ particles with momentum $\vec{k}$

(total of $\sum_{\vec{k}\in\Gamma} n(\vec{k})$ particles).

Ground state has no particles and zero energy,

$$H_0|0\rangle = 0$$

One-particle states

$$|\vec{k}\rangle \equiv \frac{1}{L^{d/2}}a^\dagger(\vec{k})|0\rangle$$

have energy

$$H_0|\vec{k}\rangle = \omega(\vec{k})|\vec{k}\rangle$$

Lowest energy level: $\omega(\vec{0}) = m$ (mass gap!) with corresponding normalized eigenstate

$$|\vec{k} = \vec{0}\rangle = \frac{a^d}{L^{d/2}} \sum_{\vec{x}\in\Omega} \left[\sqrt{\frac{m}{2}} \phi(\vec{x}) + \frac{i}{\sqrt{2m}} \pi(\vec{x}) \right] |0\rangle$$

System can be simulated with a register of N qubits for each $\vec{k} \in \Gamma$,

if mapped onto basis states $|n(\vec{k})\rangle$, $n(\vec{k}) = 0, 1, \dots, 2^N - 1$.

Alternative simulation

Working in the ϕ representation, we have

$$\pi(\vec{x}) = -\frac{i}{a^d} \frac{\partial}{\partial\phi(\vec{x})} \ , \quad a(\vec{k}) = a^d \sum_{\vec{x}\in\Omega} e^{-i\vec{k}\cdot\vec{x}} \left[\sqrt{\frac{\omega(\vec{k})}{2}} \phi(\vec{x}) + \frac{1}{\sqrt{2\omega(\vec{k})}} \frac{\partial}{\partial\phi(\vec{x})} \right]$$

Define

$$a(\vec{x}) = \frac{1}{L^d} \sum_{\vec{k}\in\Gamma} e^{i\vec{k}\cdot\vec{x}} \frac{1}{\sqrt{2\omega(\vec{k})}} a(\vec{k})$$

The ground state obeys

$$a(\vec{x})\Psi_0[\phi] = \frac{1}{2}\phi(\vec{x})\Psi[\phi] + \frac{1}{2\sqrt{-\nabla^2+m^2}}\frac{\partial\Psi[\phi]}{\partial\phi(\vec{x})} = 0 \ , \quad \forall x \in \Omega$$

therefore

$$\Psi_0[\phi] = C\exp\left\{-\frac{a^d}{2}\sum_{\vec{x}\in\Omega}\phi(\vec{x})\sqrt{-\nabla^2+m^2}\phi(\vec{x})\right\}$$

This Gaussian can be simulated by N-qubit registers at each site $\vec{x} \in \Omega$.
Other states can then be generated.
For one-particle states, introduce ancillary qubit and define

$$H_1 = a^\dagger(\vec{k})|1\rangle\langle 0| + a(\vec{k})|0\rangle\langle 1|$$

We have

$$H_1\Psi_0|0\rangle = \Psi_1|1\rangle \ , \quad H_1\Psi_1|1\rangle = \Psi_0|0\rangle$$

where $\Psi_1 = a^\dagger(\vec{k})\Psi_0$ is a one-particle state of momentum $\vec{k}$, therefore

$$e^{-iH_1\pi/2}\Psi_0|0\rangle = -i\Psi_1|1\rangle$$

Simple interacting field theory

Let us add an interaction with an external "charge" distribution $\rho(\vec{x})$, so

$$H = H_0 + H_\rho \ , \quad H_\rho = a^d \sum_{\vec{x}\in\Omega} \rho(\vec{x})\phi(\vec{x})$$

In this case, the spectrum can be calculated exactly.

We have

$$H_\rho = \frac{1}{L^d} \sum_{\vec{k}\in\Gamma} [\tilde{\rho}^*(\vec{k})a(\vec{k}) + \tilde{\rho}(\vec{k})a^\dagger(\vec{k})]$$

where

$$\tilde{\rho}(\vec{k}) = a^d \sum_{\vec{x}\in\Omega} \frac{1}{\sqrt{2\omega(\vec{k})}} \rho(\vec{x}) e^{i\vec{k}\cdot\vec{x}}$$

By defining new annihilation operator

$$b(\vec{k}) \equiv a(\vec{k}) + \frac{1}{\omega(\vec{k})}\tilde{\rho}(\vec{k})$$

we obtain

$$H = \frac{1}{L^d} \sum_{\vec{k}\in\Gamma} \left[\omega(\vec{k}) b^\dagger(\vec{k}) b(\vec{k}) - \frac{|\tilde{\rho}(\vec{k})|^2}{\omega(\vec{k})} \right]$$

The spectrum is as before, but with energy levels shifted by the new ground state energy

$$E_0 = -\frac{1}{L^d} \sum_{\vec{k}\in\Gamma} \frac{|\tilde{\rho}(\vec{k})|^2}{\omega(\vec{k})}$$

In the limit $m \to 0$, $a \to 0$, this is simply the electrostatic energy of an electric charge distribution.

The eigenstates can be found from the eigenstates of the non-interacting system by switching on H_ρ adiabatically, or by acting with the new creation operators $b^\dagger(\vec{k})$ on the new ground state. The latter is easily shown to be the coherent state (not normalized)

$$|0\rangle_\rho = \exp\left\{ -\frac{1}{L^d} \sum_{\vec{k}\in\Gamma} \frac{\tilde{\rho}(\vec{k})}{\omega(\vec{k})} a^\dagger(\vec{k}) \right\} |0\rangle$$

Quartic interaction

Hamiltonian

$$H = a^d \sum_{\vec{x}\in\Omega} \left(\frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}\phi(\vec{x})(-\nabla^2 + m_0^2)\phi(\vec{x}) + \frac{\lambda_0}{4!}\phi^4(\vec{x}) \right)$$

To prepare an initial state, start with the ground state of H_0, and adiabatically evolve it with $H(t/\tau)$, where $H(0) = H_0$ and $H(1) = H$.

One ends up with the ground state of H, if τ is long enough. The minimum τ is determined by the mass gap m_0.

♣ What to do if $m_0 = 0$?

The Path. Need to determine $H(s)$ $(0 \leq s \leq 1)$. Define

$$H(s) = a^d \sum_{\vec{x}\in\Omega} \left(\frac{1}{2}\pi^2(\vec{x}) + \frac{1}{2}\phi(\vec{x})(-\nabla^2 + m_0^2(s))\phi(\vec{x}) + \frac{\lambda_0(s)}{4!}\phi^4(\vec{x}) \right)$$

Simple choice: $m_0^2(s) = m_0^2 \quad , \quad \lambda_0(s) = s\lambda_0$

Problem: this choice does not span the entire physical parameter space.

► m_0^2 is not the physical mass. In fact, we *can* have $m_0^2 < 0$.

Solution: Use perturbation theory for an educated guess of the path.

Perturbation theory

Write $H = H_0 + H_I$, where unperturbed

$$H_0 = \frac{a^d}{2} \sum_{\vec{x}\in\Omega} \left(\pi^2(\vec{x}) + \phi(\vec{x})(-\nabla^2 + m^2)\phi(\vec{x})\right)$$

(m^2 is physical mass), and interaction Hamiltonian (perturbation)

$$H_I = a^d \sum_{\vec{x}\in\Omega} \left(\frac{\delta_m}{2}\phi^2(\vec{x}) + \frac{\lambda_0}{4!}\phi^4(\vec{x})\right)$$

mass counterterm: $\delta_m = m_0^2 - m^2$.

A (somewhat involved) calculation of the mass gap yields

$$m^2a^2 = \begin{cases} m_0^2a^2 + \frac{\lambda_0 a^2}{8\pi}\left(1 - \frac{\lambda_0 a^2}{8\pi m_0^2 a^2}\right) \log \frac{64}{m_0^2 a^2} - \frac{(\lambda_0 a^2)^2}{384 m_0^2 a^2} + \ldots & , \quad d = 1 \\ m_0^2a^2 + \frac{A_2}{16\pi^2}\left(1 - \frac{\lambda_0 a}{16\pi m_0 a}\right)\lambda_0 a + \frac{\lambda_0^2 a^2}{48} \log m_0^2 a^2 + \ldots & , \quad d = 2 \\ m_0^2a^2 + \frac{A_3}{32\pi^3}\left(1 + \frac{\lambda_0}{32\pi^2}\log m_0^2 a^2\right)\lambda_0 - \frac{B_3}{1536\pi^7}\lambda_0^2 + \ldots & , \quad d = 3 \end{cases}$$

where (one-loop contribution)

$$A_d = \int \cdots \int_{-\pi}^{\pi} \frac{d^d k}{2\sqrt{\sum_{i=1}^{d} \sin^2 \frac{k_i}{2}}}$$

and (two-loop contribution)

$$B_3 = \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \frac{1}{\sqrt{\alpha(1-\alpha) + \beta(1-\alpha-\beta)}} \int \cdots \int_{-\pi}^{\pi} \frac{d^3k d^3k'}{\Delta^2}$$

$$\Delta = 4 \sum_{i=1}^{d} \left[\alpha \sin^2 \frac{k_i}{2} + \beta \sin^2 \frac{k_i'}{2} + (1-\alpha-\beta) \sin^2 \frac{k_i + k_i'}{2} \right]$$

Numerically, $A_2 = 25.379\ldots$, $A_3 = 112.948\ldots$.

Second-order phase transition at $m^2 = 0$, $\lambda_0 = \lambda_c(m_0^2)$.

- Symmetric phase ($\phi \to -\phi$) above curve $m^2 = 0$ (in (λ_0, m_0^2) plane).
- Symmetry breaking below curve $m^2 = 0$.
- Need to choose path entirely *above* phase transition.
  - Efficient choice for *weak* coupling λ_0, using 1st-order (one-loop) perturbative result ($0 \le s \le 1$),

$$\lambda_0(s) = s\lambda_0 \ , \quad m_0^2(s) = \begin{cases} m_0^2 + \frac{(1-s)\lambda_0}{8\pi} \log \frac{64}{m_1^2 a^2} & , \quad d = 1 \\ m_0^2 + \frac{A_2}{16\pi^2} \frac{(1-s)\lambda_0}{a} & , \quad d = 2 \\ m_0^2 + \frac{A_3}{32\pi^3} \frac{(1-s)\lambda_0}{a^2} & , \quad d = 3 \end{cases}$$

where m_1^2 is the one-loop estimate of the physical mass in $d = 1$,

$$m_1^2 = m_0^2 + \frac{\lambda_0}{8\pi} \log \frac{64}{m_1^2 a^2}$$

Works even if $m_0^2 < 0$.

– For *strong* coupling λ_0, we always have $m_0^2 > 0$, so choose

$$\lambda_0(s) = s\lambda_0 \quad , \quad m_0^2(s) = m_0^2$$

Near phase transition,

$$m^2 \sim |\lambda_0 - \lambda_c|^{2\nu} \quad , \quad d < 3$$

critical exponent ν *universal*.

Numerically: $\nu = 1$ ($d = 1$), $\nu = 0.63\ldots$ ($d = 2$).

QED

Lagrangian

$$L = \frac{a^d}{2} \sum_{\vec{x}\in\Omega} \left(\vec{E}^2 - \vec{B}^2 \right)$$

where

$$\vec{E} = -\vec{\nabla} A_0 - \partial_0 \vec{A} \ , \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

Four fields, but we know photon is transverse, therefore only two degrees of freedom are physical.

System has *gauge symmetry* (local)

$$A_0 \to A_0 - \partial_0 \chi \ , \quad \vec{A} \to \vec{A} + \vec{\nabla} \chi$$

Fix gauge by imposing

$$\partial_0 A_0 + \vec{\nabla} \cdot \vec{A} = 0$$

Introduce arbitrary parameter $\lambda > 0$, and modify Lagrangian

$$L_\lambda = \frac{a^d}{2} \sum_{\vec{x}\in\Omega} \left[\vec{E}^2 - \vec{B}^2 - \lambda (\partial_0 A_0 + \vec{\nabla} \cdot \vec{A})^2 \right]$$

Conjugate momenta to fields $(A_0, \vec{A})$,

$$\pi_0(\vec{x}) = \frac{1}{a^d}\frac{\partial L_\lambda}{\partial(\partial_0 A_0(\vec{x}))} = -\lambda(\partial_0 A_0 + \vec{\nabla}\cdot\vec{A}) \ , \ \ \pi_i(\vec{x}) = \frac{1}{a^d}\frac{\partial L_\lambda}{\partial(\partial_0 A_i(\vec{x}))} = -E_i$$

Notice $\pi_0 = 0$, if $\lambda = 0$ (constrained system).

No physical results should depend on λ.

- For simplicity, set $\lambda = 1$ (*Feynman* gauge).

Commutation relations

$$[A_0(\vec{x}), \pi_0(\vec{y})] = \frac{i}{a^d}\delta_{\vec{x},\vec{y}} \ , \quad [A_i(\vec{x}), \pi_j(\vec{y})] = \frac{i}{a^d}\delta_{ij}\delta_{\vec{x},\vec{y}}$$

Hamiltonian

$$\begin{aligned} H &= a^d \sum_{\vec{x}\in\Omega} \left[-\pi_0\partial_0 A_0 + \vec{\pi}\cdot\partial_0\vec{A}\right] - L_\lambda \\ &= \frac{a^d}{2}\sum_{\vec{x}\in\Omega}\left[-(\pi_0 + \vec{\nabla}\cdot\vec{A})^2 + A_0\nabla^2 A_0 + (\vec{\pi} - \vec{\nabla}A_0)^2 - \vec{A}\cdot\nabla^2\vec{A}\right] \end{aligned}$$

Hints of trouble:

- wrong sign of kinetic term $(\pi_0 + \dots)^2$, and in $\pi_0 = -\partial_0 A_0 + \dots$.

As in scalar case, attempt to define "annihilation" operator for A_0,

$$a_0(\vec{k}) = a^d \sum_{\vec{x}\in\Omega} e^{-i\vec{k}\cdot\vec{x}} \left[\sqrt{\frac{\omega(\vec{k})}{2}} A_0(\vec{x}) + \frac{i}{\sqrt{2\omega(\vec{k})}} \left(\pi_0(\vec{x}) + \vec{\nabla}\cdot\vec{A}(\vec{x}) \right) \right]$$

where $\omega(\vec{k}) = \frac{2}{a}\sqrt{\sum_{i=1}^{d} \sin^2 \frac{k_i a}{2}}$. We have

$$H a_0^\dagger(\vec{k})|0\rangle = -\omega(\vec{k}) a_0^\dagger(\vec{k})|0\rangle$$

Negative energy!

$\therefore$ We must define $a_0(\vec{k})$ to be the creation operator!

$$|\vec{k}, 0\rangle = \frac{1}{L^{d/2}} a_0(\vec{k})|0\rangle$$

But then $\langle \vec{k}, 0|\vec{k}, 0\rangle = -1$. Negative norm state!

- Need to restrict to transverse polarizations and reject unphysical states by imposing gauge fixing condition $\partial_0 A_0 + \vec{\nabla}\cdot\vec{A} = 0$.
  We shall do this on the average:

$$\langle \Psi|\pi_0|\Psi\rangle = 0$$

Introduce annihilation operators for $\vec{A}$,

$$\vec{a}(\vec{k}) = a^d \sum_{\vec{x}\in\Omega} e^{-i\vec{k}\cdot\vec{x}} \left[\sqrt{\frac{\omega(\vec{k})}{2}} \vec{A}(\vec{x}) + \frac{i}{\sqrt{2\omega(\vec{k})}} \left(\vec{\pi}(\vec{x}) - \vec{\nabla} A_0(\vec{x}) \right) \right]$$

Hamiltonian (normal-ordered)

$$H = \sum_{\vec{k}\in\Gamma} \mathcal{H}(\vec{k}) \ , \quad \mathcal{H}(\vec{k}) = \frac{1}{L^d} \omega(\vec{k}) \left[\vec{a}^\dagger(\vec{k}) \cdot \vec{a}(\vec{k}) - a_0(\vec{k}) a_0^\dagger(\vec{k}) \right]$$

Consider the general one-particle state (not normalized)

$$|\Psi\rangle = \left(\zeta_0 a_0(\vec{k}) + \vec{\zeta} \cdot \vec{a}^\dagger(\vec{k}) \right) |0\rangle$$

We have

$$\langle \Psi | \pi_0(\vec{x}) | \Psi \rangle \propto \omega(\vec{k}) \zeta_0 - \frac{2}{a} \sum_{i=1}^{d} \sin \frac{k_i a}{2} \zeta_i = 0$$

$\Rightarrow \zeta_0$ constrained.

Positive norm!

$$\langle\Psi|\Psi\rangle \sim -|\zeta_0|^2 + |\vec{\zeta}|^2 = |\vec{\zeta}|^2 - \frac{|\vec{\kappa}\cdot\vec{\zeta}|^2}{\vec{\kappa}^2} \geq 0$$

where $\kappa_i = \frac{2}{a}\sin\frac{k_i a}{2}$.

Zero norm when $\vec{\zeta} = \vec{\kappa}$, $\zeta_0 = \omega(\vec{k})$.

- Longitudinal polarization.
- Redundancy: if $|\Psi_0\rangle$ has zero norm, then $|\Psi\rangle$ and $|\Psi\rangle + |\Psi_0\rangle$ describe <u>same</u> physical system (due to gauge invariance).
- 2 physical degrees of freedom!

◇ ◇ ◇

Addition of fixed charge (current) is handled as in the case of a scalar field.

OUTLOOK

- Quantum computation of high energy scattering amplitudes is faster than any classical algorithm (lattice field theory).
- What is the computational power of our Universe (QFTs)?
- ***Wilson*** discovered deep insights (*renormalization group*) in QFTs thinking about simulations on classical computers.
  - What insights will we gain with QC?
- Can QFC (quantum field computation) go beyond QC?
- Can we understand quantum gravity better with QC, or by thinking about information loss into a black hole?